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Transverse Intersection of Invariant Manifolds in Singular Systems

Flaviano Battelli[1]

Dipartimento di Matematica, Facoltà di Ingegneria-Università, Via Brecce Bianche, 1, 60131 Ancona, Italy

and

Kenneth J. Palmer[2]

Centre of Financial Studies, University of Melbourne, Victoria 3010, Australia

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1. INTRODUCTION

Consider the singularly perturbed system:

$$\begin{aligned} \dot{x} &= \varepsilon f(x, y, \varepsilon) \\ \dot{y} &= g(x, y, \varepsilon), \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$ and $f(x, y, \varepsilon)$, $g(x, y, \varepsilon)$ are C^{r+2}-functions in their arguments bounded with their derivatives, $r \geqslant 2$. We suppose that for any $\alpha \in \mathbf{R}^m$, the equation

$$g(\alpha, y, 0) = 0$$

has solutions $y = v_{\pm}(\alpha)$, that may coincide, such that the partial derivative of g with respect to y, $g_y(x, y, \varepsilon)$ (in this paper we denote by ϕ_x, ϕ_y, ϕ_ε the partial derivatives of the function $\phi(x, y, \varepsilon)$ with respect to x, y, ε respectively) satisfies the following eigenvalue condition:

(i) for any $\alpha \in \mathbf{R}^m$ the matrices $g_y(\alpha, v_-(\alpha), 0)$ and $g_y(\alpha, v_+(\alpha), 0)$ have the same number p (respectively $q = n - p$) of eigenvalues with positive

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(respectively negative) real parts, counted with multiplicities. Also we assume that the real parts of these eigenvalues are bounded below, in absolute value, by a positive constant independent of α.

Next we assume that for some $\alpha_0 \in \mathbf{R}^m$

(ii) the equation

$$\dot{y} = g(\alpha_0, y, 0)$$

has a solution $y_0(t)$ heteroclinic to the fixed points $v_-(\alpha_0)$, $v_+(\alpha_0)$, that is, $y_0(t) \to v_+(\alpha_0)$ as $t \to \infty$ and $y_0(t) \to v_-(\alpha_0)$ as $t \to -\infty$, such that

(iii) $\dot{y}_0(t)$ is the unique bounded solution of the linear variational system

$$\dot{y} = g_y(\alpha_0, y_0(t), 0)\, y \tag{2}$$

up to a scalar multiple.

Passing from

$$\dot{y} = g(\alpha, y, 0), \qquad \alpha \in \mathbf{R}^m$$

to (1), the normally hyperbolic manifolds $y = v_\pm(\alpha)$ perturb to global centre manifolds $y = v_\pm(x, \varepsilon)$ for equation (1) with their associated centre–stable and centre–unstable manifolds. In some previous papers ([3], [11], [12]), it has been shown in some special cases of system (1) that if an appropriate Melnikov function has a simple zero, then (1) has a solution $(x(t, \varepsilon),\ y(t, \varepsilon))$ that lies on the intersection of the centre–stable manifold of $v_+(x, \varepsilon)$ and the centre–unstable manifold of $v_-(x, \varepsilon)$. Moreover

$$|y(t, \varepsilon) - v^\pm(x(t, \varepsilon), \varepsilon)| \to 0 \qquad \text{as} \quad t \to \pm\infty \text{ uniformly in } \varepsilon \tag{3}$$

and

$$(x(t, 0), y(t, 0)) = (\alpha_0, y_0(t)). \tag{4}$$

(Actually, in the general case of (1), the fact that equation (3) holds has not to our knowledge been proved in the literature. We propose to remedy this situation in a paper [4] in preparation.) The purpose of this paper is to prove some general results on the transversality of these intersections.

Let us explain more why we want to consider this problem. Systems of the kind (1) arise many situations. One comes from nonlinear oscillations, where the equations are slowly varying in time, such as equations of the form

$$\ddot{z} + \Phi(\varepsilon t, z, \varepsilon) = 0, \tag{5}$$

where for each fixed x the "frozen" system

$$\ddot{z}+\Phi(x, z, 0)=0$$

has equilibrium solutions $z=w_+(x)$, $z=w_-(x)$ such that there is a positive constant δ with

$$\frac{\partial\Phi}{\partial z}(x, w_+(x)) \leqslant -\delta, \qquad \frac{\partial\Phi}{\partial z}(x, w_-(x)) \leqslant -\delta.$$

These conditions imply that the equilibria are saddle points. Next it is supposed that for some α_0, the equation

$$\ddot{z}+\Phi(\alpha_0, z, 0)=0$$

has a solution $z_0(t)$ connecting the two saddle points. Then hypotheses (i), (ii), (iii) are satisfied for the system

$$\begin{aligned} \dot{x} &= \varepsilon \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -\Phi(x, y_1, \varepsilon) \end{aligned} \tag{6}$$

with $v_\pm(\alpha)=(w_\pm(\alpha), 0)$, $p=q=1$ and $y_0(t)=(z_0(t), \dot{z}_0(t))$.

A special case of this situation is the equation considered in Kurland–Levi [9]

$$\ddot{z}+z(z-p(x))(1-z)=0,$$

where $p(x)$ is a periodic function for which $p(\alpha_0)=1/2$ and $p'(\alpha_0)\neq 0$. In this case $w_+(x)=1$ and $w_-(x)=0$. Kurland and Levi show that for small positive ε, the period map for

$$\ddot{z}+z(z-p(\varepsilon t))(1-z)=0$$

has a transversal heteroclinic point.

Another case is that considered by Cherry [5] and Palmer [11]

$$\ddot{z}+p(\varepsilon t)\, g(z)=0,$$

where p is a positive periodic function with a nondegenerate turning point and the conservative system

$$\ddot{z}+g(z)=0$$

has a pair w_+, w_- of saddle points with connecting orbit. In this case, $w_\pm(\alpha) = w_\pm$ and α_0 can be any real number. Palmer [11] shows that for small positive ε, the period map for

$$\ddot{z} + p(\varepsilon t)\, g(z) = 0$$

has a transversal heteroclinic point (homoclinic when $w_+ = w_-$). In the homoclinic case this implies chaos so that this result yields a simple method of constructing chaotic systems.

Still another case is that considered by Battelli and Palmer [3]

$$\ddot{z} + g(z) = \varepsilon^2 p(\varepsilon t),$$

where p is a periodic function and the conservative system

$$\ddot{z} + g(z) = 0$$

has a pair w_+, w_- of saddle points with connecting orbit. Again, in this case, $w_\pm(\alpha) = w_\pm$ and α_0 can be any real number. Battelli and Palmer show that for small positive ε, the period map for

$$\ddot{z} + g(z) = \varepsilon^2 p(\varepsilon t)$$

has a transversal heteroclinic point when $w_+ \neq w_-$ and p has a nondegenerate turning point, and has a transversal homoclinic point when $w_+ = w_-$ and the derivative p' has a nondegenerate turning point.

It has been a long term project of ours to develop a theory which unifies these results. A first step was the paper of Battelli [1]. This proved a general result for equation (1), special cases of which yielded the existence results in the three cases just described. The basic insight was that the existence of heteroclinic or homoclinic orbits in the three cases was equivalent to proving that the centre stable and centre unstable manifolds for the corresponding system (1) intersect. However the transversality was not proved and this is what we do in this paper. It is important to point out that there are two kinds of transversality here. The first is that the centre stable and centre unstable manifolds intersect transversally. However, a second kind of transversality is needed here. In order to explain this second stronger kind of transversality, let us note that the centre-stable and centre-unstable manifolds are foliated by leaves called (strongly) stable and (strongly) unstable manifolds of $v_+(x, \varepsilon)$, $v_-(x, \varepsilon)$ ([7]). Each leaf corresponds to a point on the centre manifold such that solutions starting on the leaf are asymptotic to the solution starting at the associated point on the centre manifold. The stronger kind of transversality occurs if the centre-stable manifold of $v_+(x, \varepsilon)$ intersects transversally the unstable manifold of

$v_-(x, \varepsilon)$ or if the stable manifold of $v_+(x, \varepsilon)$ intersects transversally the centre-unstable manifold of $v_-(x, \varepsilon)$.

Systems of the general form (1) have been studied by many authors as, for example, Fenichel [7], Jones [8], Lin [10], Sakamoto [12] and Szmolyan [13]. We postpone discussion of the relation of their work to ours till later in the introduction.

In this paper we consider three situations corresponding to the three cases described above. The simplest case, which we refer to as the *Kurland–Levi* case (conf. [1]), occurs when the centre-stable and centre-unstable manifolds for the unperturbed system

$$\dot{x} = 0$$

$$\dot{y} = g(x, y, 0)$$

intersect transversally along the solution $(\alpha_0, y_0(t))$. As essentially shown in [13], this happens if and only if the *Kurland–Levi* condition

$$\Delta_0'(\alpha_0) \neq 0,$$

where

$$\Delta_0(\alpha) = \int_{-\infty}^{\infty} \psi^*(t)\, g(\alpha, y_0(t), 0)\, \mathrm{d}t$$

holds. Here $\psi(t)$ is, up to a scalar multiple, the unique nonzero bounded solution of the equation adjoint to (2). In this case, of course, for ε sufficiently small, the centre-stable and centre-unstable manifolds for the perturbed system will also intersect transversally. However, as we shall see, it turns out that if a stronger condition is satisfied, the centre-stable manifold of the perturbed system intersects the unstable manifold transversally even though this is clearly not true for the unperturbed system. This explains the phenomenon in Kurland–Levi [9] that the Poincaré map for the equation

$$\ddot{x} + x(x - p(\varepsilon t))(1 - x) = 0$$

has a *transversal* heteroclinic point only when $\varepsilon \neq 0$, even though in the unperturbed system the centre-stable and centre-unstable manifolds intersect transversally. This is why we believe that the second kind of transversality studied in this paper might have notable influence in the applications.

It is more complicated when the centre-stable and centre-unstable manifolds for the unperturbed system do not intersect transversally along $(\alpha_0, y_0(t))$. In what we call the *Cherry* case (conf. [5] and [11]), the equation

$$\dot{y} = g(\alpha, y, 0)$$

has a heteroclinic or homoclinic solution for any α in $\mathbf{R}^m$ near α_0 such that both the variational system

$$\dot{y} = g_y(\alpha, y_0(t, \alpha), 0)\, y \tag{7}$$

and its adjoint

$$\dot{y} = -g_y^*(\alpha, y_0(t, \alpha), 0)\, y$$

have a one-dimensional space of bounded solutions spanned, respectively, by $\dot{y}_0(t, \alpha)$ and $\psi(t, \alpha)$ with $y_0(t) = y_0(t, \alpha_0)$. Moreover these functions and their derivatives with respect to α exist and decay exponentially to zero as $|t| \to \infty$.

Differentiating (7) with respect to α we see that $y_{0\alpha}(t, \alpha)$ is a bounded solution of

$$\dot{y} = g_y(\alpha, y_0(t, \alpha), 0)\, y + g_x(\alpha, y_0(t, \alpha), 0). \tag{8}$$

Hence

$$\begin{aligned}\frac{d}{dt}[\psi^*(t, \alpha)\, y_{0\alpha}(t, \alpha)] &= -\psi^*(t, \alpha)\, g_y(\alpha, y_0(t, \alpha), 0)\, y_{0\alpha}(t, \alpha) \\ &\quad + \psi^*(t, \alpha)[g_y(\alpha, y_0(t, \alpha), 0)\, y_{0\alpha}(t, \alpha) \\ &\quad + g_x(\alpha, y_0(t, \alpha), 0)] \\ &= \psi^*(t, \alpha)\, g_x(\alpha, y_0(t, \alpha), 0)\end{aligned}$$

and so

$$\int_{-\infty}^{+\infty} \psi^*(t, \alpha)\, g_x(\alpha, y_0(t, \alpha), 0)\, \mathrm{d}t = \int_{-\infty}^{+\infty} \frac{d}{dt}[\psi^*(t, \alpha)\, y_{0\alpha}(t, \alpha)]\, \mathrm{d}t = 0 \tag{9}$$

for any $\alpha \in \mathbf{R}$, because $y_{0\alpha}(t, \alpha)$ is bounded and $\psi(t, \alpha)$ tends to zero exponentially as $t \to \pm\infty$. Hence the Kurland–Levi condition holds along none of the solutions $(\alpha, y_0(t, \alpha))$. This is consistent with the fact that the Kurland–Levi condition implies transverse intersection between center-stable and center-unstable manifolds even for $\varepsilon = 0$. Then it turns out that the correct Melnikov function for the Cherry case is

$$\Delta_1(\alpha) = \int_{-\infty}^{+\infty} \psi^*(t, \alpha)[g_\varepsilon(\alpha, y_0(t, \alpha), 0) - y_{0\alpha}(t, \alpha)\, f(\alpha, y_0(t, \alpha), 0)]\, \mathrm{d}t \tag{10}$$

which can be also written as (see [2, 11])

$$\Delta_1(\alpha) = \int_{-\infty}^{+\infty} \psi^*(t, \alpha)[g_x(\alpha, y_0(t, \alpha), 0)\, p(t, \alpha) + g_\varepsilon(\alpha, y_0(t, \alpha), 0)]\, dt, \tag{11}$$

where

$$p(t, \alpha) = \int_0^t f(\alpha, y_0(s, \alpha), 0)\, ds. \tag{12}$$

Finally, in the third case which we refer to as the *Duffing* case (conf. [3]), we have

$$g(x, y, \varepsilon) = G(y) + \varepsilon^2 h(x, y, \varepsilon)$$

and the equation

$$\dot{y} = G(y)$$

has a heteroclinic solution $y_0(t)$. Since now $y_0(t)$ and $\psi(t)$ are independent of α and the perturbation of $\dot{y} = G(y)$ is of order $O(\varepsilon^2)$, we see from (10) that $\Delta_1(\alpha)$ is identically zero. In fact the correct Melnikov function in this case is given by:

$$\Delta_2(\alpha) = \int_{-\infty}^{+\infty} \psi^*(t)\, h(\alpha, y_0(t), 0)\, dt.$$

Now we summarise the main results of the paper. We prove first that in the Kurland–Levi, Cherry and Duffing cases the Melnikov conditions implying that the centre-stable and centre-unstable manifolds intersect also imply that these intersections are transversal. Then we show in all three cases that the centre-stable and unstable manifolds intersect transversally along the solution $(x(t, \varepsilon), (y(t, \varepsilon))$ provided that the stronger condition

$$\Delta_i'(\alpha_0)\, f(\alpha_0, v_-(\alpha_0), 0) \neq 0 \tag{13}$$

holds, and similarly that the centre-unstable and stable manifolds intersect transversally provided that

$$\Delta_i'(\alpha_0)\, f(\alpha_0, v_+(\alpha_0), 0) \neq 0 \tag{14}$$

holds.

Let us now discuss the relationship between our results and some in the literature. Sakamoto [12] seems to be proving something very close to what we have proved but only in the Kurland–Levi case. Szmolyan [13] considers the existence of orbits which are in the intersection of the stable and unstable manifolds of particular invariant sets in the centre manifolds. In particular, in the case where the invariant set is a periodic orbit, the same conditions as ours in the Kurland–Levi case ensure the transversal intersection of the stable and unstable manifolds of the periodic orbit. He assumes that the transversality already obtains in the unperturbed problem and does not consider more degenerate cases like the Cherry and Duffing cases. Our conditions occur in Lin's work [10] on transition layers without their geometric significance being realised. Hypothesis (H2) in [10, p. 326] is just the Kurland–Levi condition $\Delta_0'(\alpha_0) \neq 0$ and hypothesis (H3) is just Eqs. (13) and (14). Lin does not consider the more degenerate cases. However, in his more geometric approach, Jones [8] does consider more degenerate cases but his hypotheses are formulated in a different fashion from ours and it is difficult to make comparisons.

Throughout this paper we will assume that the Melnikov function associated to the given equation has a simple zero at $\alpha = \alpha_0$. This fact implies that system (1) has a solution $(x(t, \varepsilon), y(t, \varepsilon))$ satisfying (3) and (4) which lies in the intersection of the centre-stable and centre-unstable manifolds. Moreover, from [1, Propositions 1, 2], $(x(t, \varepsilon), y(t, \varepsilon))$ satisfies also the following condition (see also [4]). Let $z(t, \varepsilon)$ be either $x(t, \varepsilon)$ or $y(t, \varepsilon)$. Then

for given $\sigma > 0$, there exist $\varepsilon_0 > 0$, $N \geqslant 1$ such that $z(t, \varepsilon)$ is C^{r+1} and for $k = 1, \ldots, r+1$, $|\varepsilon|, |\varepsilon_1|, |\varepsilon_2| \leqslant \varepsilon_0$:

$$\left| \frac{\partial^k z(t, \varepsilon)}{\partial \varepsilon^k} \right| \leqslant N e^{k\sigma t}$$
$$\left| \frac{\partial^k z(t, \varepsilon_2)}{\partial \varepsilon^k} - \frac{\partial^k z(t, \varepsilon_1)}{\partial \varepsilon^k} \right| \leqslant N e^{(k+1)\sigma t} |\varepsilon_2 - \varepsilon_1|. \tag{15}$$

Actually, as stated in [1] these solutions depend also on other $m-1$ parameters. However we are not interested in this dependence so we omit considering it.

Studying the transversality is equivalent to studying the existence of solutions of the variational system

$$\begin{aligned} \dot{x} &= \varepsilon f_x(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)\, x + \varepsilon f_y(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)\, y \\ \dot{y} &= g_x(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)\, x + g_y(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)\, y \end{aligned} \tag{16}$$

which have special boundedness properties at $\pm\infty$. We explain this more fully in the next section, where we also prove our fundamental lemmas. The results just stated are then proved in the following two sections.

2. FUNDAMENTAL LEMMAS

We are supposing that the Melnikov function associated to the given problem has a simple zero at the point α_0. Thus, for ε sufficiently small we have solutions $(x(t, \varepsilon), y(t, \varepsilon))$ of (1) satisfying (3), (4) and (15). First we use (3) to show that the linear equation

$$\dot{y} = g_y(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)\, y$$

has certain exponential dichotomy properties. To begin, observe that (i) and Proposition 1 of Lecture 4 in Coppel [6] imply for ε sufficiently small and any α that the systems

$$\dot{y} = g_y(\alpha, v_\pm(\alpha, \varepsilon), \varepsilon)\, y$$

have exponential dichotomies on $\mathbf{R}$ with projections of rank q and constants and exponents independent of α and ε. Next, since $x(t, \varepsilon)$ is slowly varying, we know from Proposition 1 of Lecture 6 in Coppel [6] that for ε sufficiently small the systems

$$\dot{y} = g_y(x(t, \varepsilon), v_\pm(x(t, \varepsilon), \varepsilon), \varepsilon)\, y$$

have exponential dichotomies on $\mathbf{R}$ with projections of the same rank and constants and exponents independent of ε. Then, owing to (3), Proposition 1 of Lecture 4 in Coppel [6] again and also page 13 of Lecture 2 in Coppel [6], it follows that the system $\dot{y} = g_y(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)\, y$ has exponential dichotomies on both $\mathbf{R}^+$ and $\mathbf{R}^-$ with projections of the same rank and constants and exponents independent of ε. This means that there exist projections $P_+(\varepsilon)$, $P_-(\varepsilon)$ of rank q such that the fundamental matrix $Y(t, \varepsilon)$ of

$$\dot{y} = g_y(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)\, y,$$

with $Y(0, \varepsilon) = \mathbf{I}$, the identity matrix, satisfies:

$$\begin{aligned} \|Y(t, \varepsilon)\, P_+(\varepsilon)\, Y^{-1}(s, \varepsilon)\| &\leqslant k e^{-\delta(t-s)}, & &\text{if } 0 \leqslant s \leqslant t \\ \|Y(t, \varepsilon)(\mathbf{I} - P_+(\varepsilon))\, Y^{-1}(s, \varepsilon)\| &\leqslant k e^{\delta(t-s)}, & &\text{if } 0 \leqslant t \leqslant s \end{aligned}$$

and

$$\begin{aligned} \|Y(t, \varepsilon)\, P_-(\varepsilon)\, Y^{-1}(s, \varepsilon)\| &\leqslant k e^{-\delta(t-s)}, & &\text{if } s \leqslant t \leqslant 0 \\ \|Y(t, \varepsilon)(\mathbf{I} - P_-(\varepsilon))\, Y^{-1}(s, \varepsilon)\| &\leqslant k e^{\delta(t-s)}, & &\text{if } t \leqslant s \leqslant 0. \end{aligned}$$

Moreover the constants $k \geqslant 1$, $\delta > 0$ in the dichotomy can be taken independent of ε.

Now we know that $x(t, \varepsilon)$ and $y(t, \varepsilon)$) satisfy condition (15). It follows that the coefficient matrices $f_x(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)$, etc. in (16) have similar properties. Then it is a consequence of [11] that the projections $P_\pm(\varepsilon)$ can be chosen C^{r+1} in ε and such that $\mathscr{N}P_+(\varepsilon)$, $\mathscr{R}P_-(\varepsilon)$ are independent of ε. To simplify the notation we will write $P_\pm = P_\pm(0)$.

Our key observation (see [4]) is that when σ and β satisfy $0 < \sigma < \beta < \delta$ then for ε sufficiently small, the tangent space to the centre-stable manifold at $(x(t, \varepsilon), y(t, \varepsilon))$ is just the the subspace of initial values of solutions of (16) the norms of which are bounded on $\mathbf{R}_+$ when multiplied by $e^{-\sigma |t|}$. Next the tangent space to the stable manifold (or, more precisely, the leaf of the stable foliation) at $(x(t, \varepsilon), y(t, \varepsilon))$ is just the subspace of initial values of solutions of (16) the norms of which are bounded on $\mathbf{R}_+$ when multiplied by $e^{\beta |t|}$. Analogous statements hold for the centre-unstable and unstable manifolds. In fact, we will assume $(r+3)\,\sigma < \beta < \delta$.

Hence we are led to consider a linear system of differential equations like:

$$\begin{aligned} \dot{x} &= \varepsilon A(t, \varepsilon)\, x + \varepsilon B(t, \varepsilon)\, y \\ \dot{y} &= C(t, \varepsilon)\, x + D(t, \varepsilon)\, y. \end{aligned} \tag{17}$$

Let $M(t, \varepsilon)$ denote any of the matrices $A(t, \varepsilon)$, $B(t, \varepsilon)$, $C(t, \varepsilon)$, $D(t, \varepsilon)$. We assume that the following conditions hold:

(H1) for any $|\varepsilon| \leqslant \varepsilon_0$ the system

$$\dot{y} = D(t, \varepsilon)\, y$$

has an exponential dichotomy on $\mathbf{R}_\pm$ with constants $k \geqslant 1$ and $\delta > 0$ independent of ε and projections $P_\pm(\varepsilon)$ C^{r+1} in ε and such that $\mathscr{N}P_+(\varepsilon)$, $\mathscr{R}P_-(\varepsilon)$ are independent of ε. We set $P_\pm = P_\pm(0)$.

(H2) $M(t, \varepsilon)$ is C^{r+1} and there exist $N \geqslant 1$, $\sigma > 0$ such that $(r+3)\,\sigma < \delta$ and for $|\varepsilon| \leqslant \varepsilon_0$, $0 \leqslant k \leqslant r+1$:

$$\begin{aligned} &\left| \frac{\partial^k M(t, \varepsilon)}{\partial \varepsilon^k} \right| \leqslant N e^{k\sigma t} \\ &\left| \frac{\partial^k M(t, \varepsilon_2)}{\partial \varepsilon^k} - \frac{\partial^k M(t, \varepsilon_1)}{\partial \varepsilon^k} \right| \leqslant N e^{(k+1)\,\sigma t}\, |\varepsilon_2 - \varepsilon_1|. \end{aligned} \tag{18}$$

Let $C^0(\mathbf{R}, \mathbf{R}^n)$ be the space of continuous functions from $\mathbf{R}$ to $\mathbf{R}^n$. Then for $\beta \in \mathbf{R}$, set

$$C^0_\beta(\mathbf{R}, n) := \{y \in C^0(\mathbf{R}, \mathbf{R}^n) : \|y\|_\beta := \sup_{t \in \mathbf{R}} |y(t)|\, e^{\beta |t|} < +\infty\}.$$

$C^0_\beta(\mathbf{R}_+, n)$ and $C^0_\beta(\mathbf{R}_-, n)$ are analogously defined. Our first result concerns the existence of solutions of (17) that belong to $C^0_\beta(\mathbf{R}_-, n+m)$, β being a positive number between $(r+3)\,\sigma$ and δ.

Lemma 1 (exponential decay at $-\infty$). *Assume* (H1) *and* (H2) *and let* $(r+3)\,\sigma < \beta < \delta$. *Then there exists* $\varepsilon_0 > 0$ *such that, for* $|\varepsilon| < \varepsilon_0$, *and* $\zeta_- \in \mathcal{N}P_-$, *system* (17) *has a unique solution* $(x^-(t, \zeta_-, \varepsilon), y^-(t, \zeta_-, \varepsilon))$ *in* $C^0_\beta(\mathbf{R}_-, n+m)$ *satisfying:*

$$(\mathbf{I} - P_-)\, y^-(0, \zeta_-, \varepsilon) = \zeta_-. \tag{19}$$

Moreover $(x^-(t, \zeta_-, \varepsilon), y^-(t, \zeta_-, \varepsilon))$ *satisfies the fixed point equation:*

$$\begin{aligned}
x^-(t, \zeta_-, \varepsilon) &= \varepsilon \int_{-\infty}^{t} A(s, \varepsilon)\, x^-(s, \zeta_-, \varepsilon) + B(s, \varepsilon)\, y^-(s, \zeta_-, \varepsilon)\, \mathrm{d}s \\
y^-(t, \zeta_-, \varepsilon) &= Y(t)\, \zeta_- + \int_{-\infty}^{t} Y(t)\, P_- Y^{-1}(s) \{C(s, \varepsilon)\, x^-(s, \zeta_-, \varepsilon) \\
&\qquad + [D(s, \varepsilon) - D(s, 0)]\, y^-(s, \zeta_-, \varepsilon)\}\, \mathrm{d}s \\
&\qquad - \int_{t}^{0} Y(t)(\mathbf{I} - P_-)\, Y^{-1}(s) \{C(s, \varepsilon)\, x^-(s, \zeta_-, \varepsilon) \\
&\qquad + [D(s, \varepsilon) - D(s, 0)]\, y^-(s, \zeta_-, \varepsilon)\}\, \mathrm{d}s.
\end{aligned} \tag{20}$$

On account of linearity,

$$x^-(t, \zeta_-, \varepsilon) = [X^-(\varepsilon)\, \zeta_-](t), \quad y^-(t, \zeta_-, \varepsilon) = [Y^-(\varepsilon)\, \zeta_-](t),$$

where $(X^-(\varepsilon), Y^-(\varepsilon)) \in \mathcal{L}(\mathcal{N}P_-, C^0_\beta(\mathbf{R}_-, n+m))$. *For any* $k = 0, \ldots, r+1$, *the map*

$$\varepsilon \mapsto (X^-(\varepsilon), Y^-(\varepsilon)) \in \mathcal{L}(\mathcal{N}P_-, C^0_{\beta-(k+1)\,\sigma}(\mathbf{R}_-, n+m))$$

is C^k_{lip} *and its k-th derivative takes values in* $\mathcal{L}(\mathcal{N}P_-, C^0_{\beta-k\sigma}(\mathbf{R}_-, n+m))$.

Remark. There is a similar result for positive t. In this case we have a unique solution $(x^+(t, \zeta_+, \varepsilon), y^+(t, \zeta_+, \varepsilon))$ in $C^0_\beta(\mathbf{R}_+, n+m)$ that satisfies $P_+ y^+(0, \zeta_+, \varepsilon) = \zeta_+ \in \mathscr{R}P_+$ and the fixed point equation

$$\begin{aligned} x^+(t, \zeta_+, \varepsilon) &= -\varepsilon \int_t^{+\infty} A(s, \varepsilon)\, x^+(s, \zeta_+, \varepsilon) + B(s, \varepsilon)\, y^+(s, \zeta_+, \varepsilon)\, \mathrm{d}s \\ y^+(t, \zeta_+, \varepsilon) &= Y(t)\, \zeta_+ + \int_0^t Y(t)\, P_+ Y^{-1}(s) \{C(s, \varepsilon)\, x^+(s, \zeta_+, \varepsilon) \\ &\quad + [D(s, \varepsilon) - D(s, 0)]\, y^+(s, \zeta_+, \varepsilon)\}\, \mathrm{d}s \\ &\quad - \int_t^{+\infty} Y(t)(\mathbf{I} - P_+)\, Y^{-1}(s) \{C(s, \varepsilon)\, x^+(s, \zeta_+, \varepsilon) \\ &\quad + [D(s, \varepsilon) - D(s, 0)]\, y^-(s, \zeta_+, \varepsilon)\}\, \mathrm{d}s. \end{aligned} \tag{21}$$

and a smoothness property similar to $(x^-(t, \zeta_-, \varepsilon), y^-(t, \zeta_-, \varepsilon))$.

Our second result concerns the existence of solutions of (17) that belong to the space $C^0_{-\sigma}(\mathbf{R}_+, n+m)$.

LEMMA 2 (bounded growth at $+\infty$). *Assume* (H1) *and* (H2). *Then there exists* $\varepsilon_0 > 0$ *such that, for* $|\varepsilon| < \varepsilon_0$, $\xi \in \mathbf{R}^m$ *and* $\zeta_+ \in \mathscr{R}P_+$, *system* (17) *has a unique solution* $(x^+(t, \xi, \zeta_+, \varepsilon), y^+(t, \xi, \zeta_+, \varepsilon)) \in C^0_{-\sigma}(\mathbf{R}_+, n+m)$ *satisfying:*

$$x^+(0, \xi, \zeta_+, \varepsilon) = \xi, \qquad P_+ y^+(0, \xi, \zeta_+, \varepsilon) = \zeta_+.$$

Moreover $(x^+(t, \xi, \zeta_+, \varepsilon), y^+(t, \xi, \zeta_+, \varepsilon))$ *is a solution of the fixed point equation:*

$$\begin{aligned} x^+(t, \xi, \zeta_+, \varepsilon) &= \xi + \varepsilon \int_0^t A(s, \varepsilon)\, x^+(s, \xi, \zeta_+, \varepsilon) + B(s, \varepsilon)\, y^+(s, \xi, \zeta_+, \varepsilon)\, \mathrm{d}s \\ y^+(t, \xi, \zeta_+, \varepsilon) &= Y(t)\, \zeta_+ + \int_0^t Y(t)\, P_+ Y^{-1}(s) \{C(s, \varepsilon)\, x^+(s, \xi, \zeta_+, \varepsilon) \\ &\quad + [D(s, \varepsilon) - D(s, 0)]\, y^+(s, \xi, \zeta_+, \varepsilon)\}\, \mathrm{d}s \\ &\quad - \int_t^{\infty} Y(t)(\mathbf{I} - P_+)\, Y^{-1}(s) \{C(s, \varepsilon)\, x^+(s, \xi, \zeta_+, \varepsilon) \\ &\quad + [D(s, \varepsilon) - D(s, 0)]\, y^+(s, \xi, \zeta_+, \varepsilon)\}\, \mathrm{d}s. \end{aligned} \tag{22}$$

On account of linearity,

$$x^{+}(t, \xi, \zeta_{+}, \varepsilon) = [X_1^{+}(\varepsilon)\, \xi + X_2^{+}(\varepsilon)\, \zeta_{+}](t),$$
$$y^{+}(t, \xi, \zeta_{+}, \varepsilon) = [Y_1^{+}(\varepsilon)\, \xi + Y_2^{+}(\varepsilon)\, \zeta_{+}](t),$$

where $(X_1^{+}(\varepsilon), Y_1^{+}(\varepsilon)) \in \mathscr{L}(\mathbf{R}^m, C^0_{-\sigma}(\mathbf{R}_+, n+m))$ *and* $(X_2^{+}(\varepsilon), Y_2^{+}(\varepsilon)) \in \mathscr{L}(\mathscr{R}P_+, C^0_{-\sigma}(\mathbf{R}_+, n+m))$. *For any* $k = 0, \ldots, r+1$, *the maps*

$$\varepsilon \mapsto (X_1^{+}(\varepsilon), Y_1^{+}(\varepsilon)) \in \mathscr{L}(\mathbf{R}^m, C^0_{-(k+2)\sigma}(\mathbf{R}_+, n+m))$$

and

$$\varepsilon \mapsto (X_2^{+}(\varepsilon), Y_2^{+}(\varepsilon)) \in \mathscr{L}(\mathscr{R}P_+, C^0_{-(k+2)\sigma}(\mathbf{R}_+, n+m))$$

are C^k_{lip} *and their k-th derivatives take values in* $\mathscr{L}(\mathbf{R}^m, C^0_{-(k+1)\sigma}(\mathbf{R}_+, n+m))$ *and* $\mathscr{L}(\mathscr{R}P_+, C^0_{-(k+1)\sigma}(\mathbf{R}_+, n+m))$ *respectively.*

Remark. Lemma 2 has an analogue for negative t. In this case we have a solution $(x^{-}(t, \xi, \zeta_{-}, \varepsilon), y^{-}(t, \xi, \zeta_{-}, \varepsilon))$ in $C^0_{-\sigma}(\mathbf{R}_-, n+m)$ with $x^{-}(0, \xi, \zeta_{-}, \varepsilon) = \xi \in \mathbf{R}^m$, $(\mathbf{I} - P_-)\, y^{-}(0, \xi, \zeta_{-}, \varepsilon) = \zeta_{-} \in \mathscr{N}P_-$, which is linear in (ξ, ζ_-) and satisfies the fixed point equation:

$$x^{-}(t, \xi, \zeta_{-}, \varepsilon) = \xi + \varepsilon \int_0^t A(s, \varepsilon)\, x^{-}(s, \xi, \zeta_{-}, \varepsilon) + B(s, \varepsilon)\, y^{-}(s, \xi, \zeta_{-}, \varepsilon)\, \mathrm{d}s$$

$$\begin{aligned} y^{-}(t, \xi, \zeta_{-}, \varepsilon) = Y(t)\, \zeta_{-} &+ \int_{-\infty}^{t} Y(t)\, P_- Y^{-1}(s) \{C(s, \varepsilon)\, x^{-}(s, \xi, \zeta_{-}, \varepsilon) \\ &+ [D(s, \varepsilon) - D(s, 0)]\, y^{-}(s, \xi, \zeta_{-}, \varepsilon)\}\, \mathrm{d}s \\ &- \int_t^0 Y(t)(\mathbf{I} - P_-)\, Y^{-1}(s) \{C(s, \varepsilon)\, x^{-}(s, \xi, \zeta_{-}, \varepsilon) \\ &+ [D(s, \varepsilon) - D(s, 0)]\, y^{-}(s, \xi, \zeta_{-}, \varepsilon)\}\, \mathrm{d}s \end{aligned} \tag{23}$$

and that has a smoothness property similar to $(x^{+}(t, \xi, \zeta_{+}, \varepsilon), y^{+}(t, \xi, \zeta_{+}, \varepsilon))$.

The proofs of the above lemmas will be given in the appendix.

What is important to emphasise here is the fact that the derivatives of $(x(t, \varepsilon), y(t, \varepsilon))$ satisfy (15). This implies that the linear system (16) satisfies (H1) and (H2) with

$$\begin{aligned} A(t, \varepsilon) &= f_x(x(t, \varepsilon), y(t, \varepsilon), \varepsilon), \\ B(t, \varepsilon) &= f_y(x(t, \varepsilon), y(t, \varepsilon), \varepsilon), \\ C(t, \varepsilon) &= g_x(x(t, \varepsilon), y(t, \varepsilon), \varepsilon), \\ D(t, \varepsilon) &= g_y(x(t, \varepsilon), y(t, \varepsilon), \varepsilon). \end{aligned} \tag{24}$$

So Lemmas 1 and 2 hold for (16) and we can then use them to study the intersection of the perturbed center-stable ($\mathcal{M}^{cs}$) and center-unstable ($\mathcal{M}^{cu}$) or unstable ($\mathcal{M}^{u}$) manifolds along $(x(t, \varepsilon), y(t, \varepsilon))$.

3. TRANSVERSE INTERSECTION OF $\mathcal{M}^{cs}$ AND $\mathcal{M}^{cu}$

In this section we assume that, for any of the three cases considered in the introduction (Kurland–Levi, Cherry, Duffing) the corresponding Melnikov function has a simple zero at a point $\alpha = \alpha_0$. Then, as we noted in the introduction, system (1) has a solution $(x(t, \varepsilon), y(t, \varepsilon))$ satisfying (3), (4). We will show that in all three cases the centre–stable and centre–unstable manifolds intersect transversally along $(x(t, \varepsilon), y(t, \varepsilon))$ when $\varepsilon \neq 0$ is sufficiently small (also for $\varepsilon = 0$ in the Kurland–Levi case).

First note it follows from Lemma 2 and the Remark after it that system (16) has unique solutions $(x^{\pm}(t), y^{\pm}(t)) = (x^{\pm}(t, \xi, \zeta_{\pm}, \varepsilon), y^{\pm}(t, \xi, \zeta_{\pm}, \varepsilon))$ such that

$$\begin{aligned} &\sup_{t \geqslant 0} |x^{+}(t)| \, e^{-\sigma t} < \infty, \qquad &&\sup_{t \geqslant 0} |y^{+}(t)| \, e^{-\sigma t} < \infty, \\ &\sup_{t \leqslant 0} |x^{-}(t)| \, e^{\sigma t} < \infty, \qquad &&\sup_{t \leqslant 0} |y^{-}(t)| \, e^{\sigma t} < \infty \end{aligned} \tag{25}$$

satisfying respectively

$$\begin{aligned} x^{+}(0) &= \xi, \qquad & P_{+} \, y^{+}(0) &= \zeta_{+}, \\ x^{-}(0) &= \xi, \qquad & (\mathbf{I} - P_{-}) \, y^{-}(0) &= \zeta_{-}. \end{aligned} \tag{26}$$

Note that from $y(t, 0) = y_0(t)$ we deduce that $(x^{\pm}(t, \xi, \zeta_{\pm}, 0), y^{\pm}(t, \xi, \zeta_{\pm}, 0))$ is the unique solution, for $t \in \mathbf{R}_{\pm}$ respectively, of the system

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= g_y(\alpha_0, y_0(t), 0) \, y + g_x(\alpha_0, y_0(t), 0) \, x \end{aligned}$$

that satisfies conditions (25), (26). Thus

$$x^{\pm}(t, \xi, \zeta_{\pm}, 0) = \xi \tag{27}$$

and $y^{\pm}(t, \xi, \zeta_{\pm}, 0)$ is a solution in $C^0_{-\sigma}(\mathbf{R}_{\pm}, n)$ of:

$$\begin{cases} \dot{y} = g_y(\alpha_0, y_0(t), 0) \, y + g_x(\alpha_0, y_0(t), 0) \, \xi \\ P_{+} \, y^{+}(0) = \zeta_{+}, \quad (\mathbf{I} - P_{-}) \, y^{-}(0) = \zeta_{-}. \end{cases} \tag{28}$$

In particular, since $\dot{y}_0(t)$ is a bounded solution of (2), which has exponential dichotomies on $\mathbf{R}_+$, $\mathbf{R}_-$ with projections P_+, P_-,

$$y^{\pm}(t, 0, \dot{y}_0(0), 0) = \dot{y}_0(t). \tag{29}$$

We want to study the problem of the transverse intersection of $\mathscr{M}^{cs}$, which has dimension $m+q$, and $\mathscr{M}^{cu}$, which has dimension $m+p$, along the solution $(x(t, \varepsilon), y(t, \varepsilon))$. Note, as observed in the previous section, that the intersection of the tangent spaces to $\mathscr{M}^{cs}$ and $\mathscr{M}^{cu}$ at $(x(0, \varepsilon), y(0, \varepsilon))$ is the set of initial values of solutions of (16) that do not grow faster than $e^{\sigma|t|}$. Thus to prove the transversality, we need only show that the subspace of $\mathbf{R}^{n+m}$ given by

$$\{(\xi, y^+(0, \xi, \zeta_+, \varepsilon)) : y^+(0, \xi, \zeta_+, \varepsilon) = y^-(0, \xi, \zeta_-, \varepsilon)\}$$

is at most m-dimensional (actually it will then be exactly m-dimensional since it is at least m-dimensional because $q+p=n$).

Now, we define the linear operator $L(\varepsilon)$: $\mathbf{R}^m \times \mathscr{R}P_+ \times \mathscr{N}P_- \to \mathbf{R}^n$ by

$$\begin{aligned} L(\varepsilon)(\xi, \zeta_+, \zeta_-) = \int_{-\infty}^{0} & P_- Y^{-1}(t)\{C(t, \varepsilon)\, x^-(t, \xi, \zeta_-, \varepsilon) \\ & + [D(t, \varepsilon) - D(t, 0)]\, y^-(t, \xi, \zeta_-, \varepsilon)\}\, \mathrm{d}t \\ & + \int_0^{\infty} (\mathbf{I} - P_+)\, Y^{-1}(t)\{C(t, \varepsilon)\, x^+(t, \xi, \zeta_+, \varepsilon) \\ & + [D(t, \varepsilon) - D(t, 0)]\, y^+(t, \xi, \zeta_+, \varepsilon)\}\, \mathrm{d}t. \end{aligned}$$

Then, using (22) and (23), we see that

$$y^+(0, \xi, \zeta_+, \varepsilon) - y^-(0, \xi, \zeta_-, \varepsilon) = \zeta_+ - \zeta_- - L(\varepsilon)(\xi, \zeta_+, \zeta_-) \tag{30}$$

so that the equation

$$y^+(0, \xi, \zeta_+, \varepsilon) = y^-(0, \xi, \zeta_-, \varepsilon) \tag{31}$$

can be written as:

$$\zeta_+ - \zeta_- = L(\varepsilon)(\xi, \zeta_+, \zeta_-). \tag{32}$$

Note that $P_\pm$ are projections for the dichotomy of system (2) on $\mathbf{R}_\pm$. Thus, because of the exponential decay of $P_- Y^{-1}(t)$ in $t \leqslant 0$ and $(\mathbf{I} - P_+)\, Y^{-1}(t)$ in $t \geqslant 0$, and the properties of $A(t, \varepsilon)$, $B(t, \varepsilon)$, $C(t, \varepsilon)$, $D(t, \varepsilon)$, $x^{\pm}(t, \xi, \zeta_\pm, \varepsilon)$, $y^{\pm}(t, \xi, \zeta_\pm, \varepsilon)$, the map $\varepsilon \mapsto L(\varepsilon) \in \mathscr{L}(\mathbf{R}^m \times \mathscr{R}P_+ \times \mathscr{N}P_-, \mathbf{R}^n)$ is C^{r+1}. Moreover, the exponential dichotomy of (2) on both half-lines with constants k and δ, implies that $\dot{y}_0(t)$ is not only bounded on $\mathbf{R}$ but, in fact, satisfies for all $t \in \mathbf{R}$:

$$|\dot{y}_0(t)| \leqslant k\, |\dot{y}_0(0)|\, e^{-\delta |t|}.$$

Also, from hypothesis (iii), it follows that $\mathscr{R}P_+ \cap \mathscr{N}P_-$ is spanned by $\phi_0 = \dot{y}_0(0)$. Next note that the adjoint system

$$\dot{y} = -g_y(x(t,0), y(t,0), 0)^*\, y = -g_y(\alpha_0, y_0(t), 0)^*\, y \tag{33}$$

also has exponential dichotomies on $\mathbf{R}_\pm$ with constants $k \geqslant 1$, $\delta > 0$ and projections $\mathbf{I} - P^*_\pm$ and, because of conditions (i) (iii), (33) has a unique, up to a multiplicative constant, non–zero solution $\psi(t)$ which is bounded on $\mathbf{R}$ (see [11]) and, in fact, satisfies the same estimate as $\dot{y}_0(t)$. We set $\psi := \psi(0)$ and assume, without loss of generality, that $|\psi| = 1$. Note that ψ spans $\mathscr{N}P^*_+ \cap \mathscr{R}P^*_- = [\mathscr{R}P_+ + \mathscr{N}P_-]^0$, where $[V]^0$ means the annihilator of the subspace V.

Now we use a Lyapunov–Schmidt reduction to study the equation (32). Since ψ spans $[\mathscr{R}P_+ + \mathscr{N}P_-]^0$, equation (32) is equivalent to the pair of equations:

$$\begin{cases} \zeta_+ - \zeta_- = L(\varepsilon)(\xi, \zeta_+, \zeta_-) - \psi^* L(\varepsilon)(\xi, \zeta_+, \zeta_-) \cdot \psi \\ \psi^* L(\varepsilon)(\xi, \zeta_+, \zeta_-) = 0. \end{cases}$$

First we solve

$$\zeta_+ - \zeta_- = L(\varepsilon)(\xi, \zeta_+, \zeta_-) - \psi^* L(\varepsilon)(\xi, \zeta_+, \zeta_-) \cdot \psi. \tag{34}$$

Let (ξ, ζ_+, ζ_-) be a solution of equation (32) and set

$$\lambda = \langle \zeta_+, \phi_0 \rangle / \langle \phi_0, \phi_0 \rangle, \qquad \bar{\zeta}_+ = \zeta_+ - \lambda\phi_0, \qquad \bar{\zeta}_- = \zeta_- - \lambda\phi_0,$$

where, as we know, $\phi_0 = \dot{y}_0(0)$ spans $\mathscr{R}P_+ \cap \mathscr{N}P_-$. Then (34) becomes,

$$\begin{aligned} \bar{\zeta}_+ - \bar{\zeta}_- = {} & L(\varepsilon)(\xi, \bar{\zeta}_+ + \lambda\phi_0, \bar{\zeta}_- + \lambda\phi_0) \\ & - \psi^* L(\varepsilon)(\xi, \bar{\zeta}_+ + \lambda\phi_0, \bar{\zeta}_- + \lambda\phi_0) \cdot \psi. \end{aligned} \tag{35}$$

Note that $(\bar{\zeta}_+, \bar{\zeta}_-) \to \bar{\zeta}_+ - \bar{\zeta}_-$ is an invertible linear mapping from the subspace

$$\{(\bar{\zeta}_+, \bar{\zeta}_-) \in \mathscr{R}P_+ \times \mathscr{N}P_- : \langle \bar{\zeta}_+, \phi_0 \rangle = 0\}$$

onto $\mathscr{R}P_+ + \mathscr{N}P_-$ and that, from (27),

$$L(0)(0, \zeta_+, \zeta_-) = 0. \tag{36}$$

So, for ε sufficiently small, the implicit function theorem tells us that (35) has a unique solution for $\bar{\zeta}_+$, $\bar{\zeta}_-$ satisfying

$$\langle \bar{\zeta}_+, \phi_0 \rangle = 0$$

given, on account of linearity, by

$$\bar{\zeta}_\pm = H_\pm(\varepsilon)\, \xi + \omega_\pm(\varepsilon)\, \lambda,$$

where $H_\pm$ and $\omega_\pm$ are C^{r+1}. So, for ε small,

$$\zeta_\pm = \zeta_\pm(\xi, \lambda, \varepsilon) = H_\pm(\varepsilon)\, \xi + \omega_\pm(\varepsilon)\, \lambda + \lambda\phi_0$$

gives all the solutions of (34) in terms of the parameter λ. Note it follows from (36) that

$$\omega_\pm(0) = 0. \tag{37}$$

Now to obtain a solution of equation (32), we need to solve

$$\psi^* L(\varepsilon)((\xi, H_+(\varepsilon)\, \xi, H_-(\varepsilon)\, \xi) + (0, \omega_+(\varepsilon) + \phi_0, \omega_-(\varepsilon) + \phi_0)\, \lambda) = 0$$

that is,

$$h_1(\varepsilon)\, \xi + h_2(\varepsilon)\, \lambda = 0, \tag{38}$$

where

$$h_1(\varepsilon)\, \xi = \psi^* L(\varepsilon)(\xi, H_+(\varepsilon)\, \xi, H_-(\varepsilon)\, \xi) \tag{39}$$

and

$$h_2(\varepsilon) = \psi^* L(\varepsilon)(0, \omega_+(\varepsilon) + \phi_0, \omega_-(\varepsilon) + \phi_0). \tag{40}$$

This means that the intersection of the tangent spaces to $\mathscr{M}^{cs}$ and $\mathscr{M}^{cu}$ at $(x(0, \varepsilon), y(0, \varepsilon))$ is the subspace

$$\{(\xi, y^+(0, \xi, H_+(\varepsilon)\, \xi + \omega_+(\varepsilon)\, \lambda + \lambda\phi_0, \varepsilon)) : h_1(\varepsilon)\, \xi + h_2(\varepsilon)\, \lambda = 0\}.$$

Clearly this subspace will be at most m-dimensional and the transversality will follow if we can solve (38) for one of $\xi_1, \xi_2, \ldots, \xi_m, \lambda$ in terms of the others and ε. We will do this separately for the three cases considered in this paper but before doing so note that using (30) and since $\zeta_\pm = \zeta_\pm(\xi, \lambda, \varepsilon)$ satisfies (34):

$$\begin{aligned} y^+(0, \xi, \zeta_+(\xi, \lambda, \varepsilon), \varepsilon) - y^-(0, \xi, \zeta_-(\xi, \lambda, \varepsilon), \varepsilon) \\ = -\psi^* L(\varepsilon)(\xi, \zeta_+(\xi, \lambda, \varepsilon), \zeta_-(\xi, \lambda, \varepsilon)). \end{aligned} \tag{41}$$

Moreover, it follows from (36) that

$$h_2(0)=0. \tag{42}$$

3.1. *Kurland–Levi Case*

When $\varepsilon=0$,

$$L(0)(\xi,\zeta_+,\zeta_-)=\int_{-\infty}^{0} P_- Y^{-1}(t)\,C(t,0)\,x^-(t,\xi,\zeta_-,0)\,dt$$

$$+\int_0^{\infty}(\mathbf{I}-P_+)\,Y^{-1}(t)\,C(t,0)\,x^+(t,\xi,\zeta_+,0)\,\mathrm{d}t.$$

Hence, using (27),

$$\psi^*L(0)(\xi,\zeta_+,\zeta_-)=\int_{-\infty}^{\infty}\psi^*(t)\,C(t,0)\,\mathrm{d}t\cdot\xi. \tag{43}$$

So

$$h_1(0)\,\xi=\psi^*L(0)(\xi,H_+(0)\,\xi,H_-(0)\,\xi)=\int_{-\infty}^{\infty}\psi^*(t)\,C(t,0)\,\mathrm{d}t\cdot\xi; \tag{44}$$

that is,

$$h_1(0)=\int_{-\infty}^{\infty}\psi^*(t)\,C(t,0)\,\mathrm{d}t=\int_{-\infty}^{\infty}\psi^*(t)\,g_x(\alpha_0,y_0(t),0)\,\mathrm{d}t=\Delta_0'(\alpha_0) \tag{45}$$

which is not zero by the Kurland–Levi condition. Hence, $h_1(\varepsilon)\neq 0$ if ε is small and so (38) can be solved for one of the components of ξ uniquely in terms of the others, λ and ε. So we have transverse intersection of $\mathscr{M}^{cs}$ and $\mathscr{M}^{cu}$ in this case (even when $\varepsilon=0$, of course).

3.2. *Cherry Case*

In this case, it follows from (9) and (45) that

$$h_1(0)=0. \tag{46}$$

Also, from (42), we already know that $h_2(0)=0$. Hence we do not have transversality when $\varepsilon=0$. We will get transversality for $\varepsilon\neq 0$ small, if either $h_1'(0)\neq 0$ or $h_2'(0)\neq 0$. We show that

$$h_1'(0)\neq 0.$$

In order to calculate $h_1'(0)$, we use (36), (39) and (43) to get

$$\begin{aligned} h_1'(0)\,\xi &= \psi^* L'(0)(\xi, H_+(0)\,\xi, H_-(0)\,\xi) + \psi^* L(0)(0, H'_+(0)\,\xi, H'_-(0)\,\xi) \\ &= \psi^* L'(0)(\xi, H_+(0)\,\xi, H_-(0)\,\xi). \end{aligned}$$

Then from the definition of $L(\varepsilon)$ before (30) and using (27), we see that

$$\begin{aligned} \psi^* L'(0)(\xi, \zeta_+, \zeta_-) = \int_{-\infty}^{0} & \psi^*(t)[C_\varepsilon(t,0)\,\xi + C(t,0)\,x_\varepsilon^-(t, \xi, \zeta_-, 0) \\ & + D_\varepsilon(t,0)\,y^-(t, \xi, \zeta_-, 0)]\,dt \\ & + \int_0^\infty \psi^*(t)[C_\varepsilon(t,0)\,\xi + C(t,0)\,x_\varepsilon^+(t, \xi, \zeta_+, 0) \\ & + D_\varepsilon(t,0)\,y^+(t, \xi, \zeta_+, 0)]\,dt. \end{aligned} \tag{47}$$

However, equation (41) together with $H_\pm(0)\,\xi = \zeta_\pm(\xi, 0, 0)$, (43), (45) and (46) give

$$y^+(0, \xi, H_+(0)\,\xi, 0) = y^-(0, \xi, H_-(0)\,\xi, 0)$$

for all ξ. Hence

$$y(t, \xi) = \begin{cases} y^+(t, \xi, H_+(0)\,\xi, 0) & \text{if} \quad t \geqslant 0 \\ y^-(t, \xi, H_-(0)\,\xi, 0) & \text{if} \quad t < 0 \end{cases} \tag{48}$$

is a C^1 solution of the equation

$$\dot{y} = C(t, 0)\,\xi + D(t, 0)\,y$$

in $C^0_{-\sigma}(\mathbf{R}, n)$. But we know that $y_{0\alpha}(t, \alpha_0)\,\xi$ is a bounded solution of the same equation (see (8)) and so $y(t, \xi) - y_{0\alpha}(t, \alpha_0)\,\xi$ is a solution of

$$\dot{y} = D(t, 0)\,y$$

in $C^0_{-\sigma}(\mathbf{R}, n)$. However, from the exponential dichotomy properties of the last equation, $\dot{y}_0(t)$ is, up to a scalar multiple, its unique solution in

$C^0_\beta(\mathbf{R}, n)$ for any β in $-\delta < \beta < \delta$. As a consequence and using the linearity of $y(t, \xi)$ in ξ

$$y(t, \xi) = y_{0\alpha}(t, \alpha_0)\, \xi + (\mu_0^* \xi)\, \dot{y}_0(t) \tag{49}$$

for some $\mu_0 \in \mathbf{R}^m$.

Next, using (27), it is not difficult to see that

$$q(t, \xi) = \begin{cases} x_\varepsilon^+(t, \xi, H_+(0)\, \xi, 0) & \text{for} \quad t \geqslant 0 \\ x_\varepsilon^-(t, \xi, H_-(0)\, \xi, 0) & \text{for} \quad t < 0 \end{cases}$$

is a C^1 solution of

$$\begin{aligned} \dot{x} &= A(t, 0)\, \xi + B(t, 0)\, y(t, \xi) \\ x(0) &= 0. \end{aligned}$$

Hence

$$q(t, \xi) = \int_0^t [A(s, 0)\, \xi + B(s, 0)\, y(s, \xi)]\, \mathrm{d}s.$$

Next from equation (12) we get

$$p_\alpha(t, \alpha_0) = \int_0^t A(s, 0) + B(s, 0)\, y_{0\alpha}(s, \alpha_0)\, \mathrm{d}s \tag{50}$$

and then, using (49), we see that

$$\begin{aligned} q(t, \xi) &= p_\alpha(t, \alpha_0)\, \xi + (\mu_0^* \xi) \int_0^t B(s, 0)\, \dot{y}_0(s)\, ds \\ &= p_\alpha(t, \alpha_0)\, \xi + (\mu_0^* \xi)[f(\alpha_0, y_0(t), 0) - f(\alpha_0, y_0(0), 0)]. \end{aligned}$$

So, from the last equation, (49) and (47) with $\zeta_\pm = H_\pm(0)\, \xi$,

$$\begin{aligned} h_1'(0)\, \xi = \int_{-\infty}^{\infty} & \psi^*(t)\{C_\varepsilon(t, 0) + C(t, 0)\, p_\alpha(t, \alpha_0) \\ & + D_\varepsilon(t, 0)\, y_{0\alpha}(t, \alpha_0)\}\, \mathrm{d}t \cdot \xi \\ & + (\mu_0^* \xi) \int_{-\infty}^{\infty} \psi^*(t)\{C(t, 0)[f(\alpha_0, y_0(t), 0) - f(\alpha_0, y_0(0), 0)] \\ & + D_\varepsilon(t, 0)\, \dot{y}_0(t)\}\, \mathrm{d}t. \end{aligned}$$

Then, using (44) and $h_1(0)=0$,

$$h_1'(0)\,\xi=\int_{-\infty}^{\infty}\psi^*(t)\{C_\varepsilon(t,0)+C(t,0)\,p_\alpha(t,\alpha_0)$$
$$+D_\varepsilon(t,0)\,y_{0\alpha}(t,\alpha_0)\}\,\mathrm{d}t\cdot\xi$$
$$+(\mu_0^*\xi)\int_{-\infty}^{\infty}\psi^*(t)[C(t,0)\,f(\alpha_0,y_0(t),0)+D_\varepsilon(t,0)\,\dot y_0(t)]\,\mathrm{d}t. \tag{51}$$

Now we show

$$h_1'(0)=\Delta_1'(\alpha_0).$$

To this end, we differentiate equation (11), getting

$$\Delta_1'(\alpha_0)=\int_{-\infty}^{\infty}\psi_\alpha^*(t,\alpha_0)[g_\varepsilon+g_x\,p(t,\alpha_0)]\,\mathrm{d}t$$
$$+\int_{-\infty}^{\infty}\psi^*(t,\alpha_0)[g_{x\varepsilon}+g_{y\varepsilon}\,y_{0\alpha}(t,\alpha_0)+g_{xx}\,p(t,\alpha_0)$$
$$+g_{xy}\,y_{0\alpha}(t,\alpha_0)\,p(t,\alpha_0)+g_x\,p_\alpha(t,\alpha_0)]\,\mathrm{d}t,$$

where $g_x=g_x(\alpha_0,y_0(t),0)$ etc.

Now, we differentiate (1) with $(x(t,\varepsilon),y(t,\varepsilon))$ instead of (x,y) with respect to ε at $\varepsilon=0$. Using (4) we see that

$$\dot x_\varepsilon(t,0)=f(\alpha_0,y_0(t),0) \tag{52}$$

and that $y_\varepsilon(t,0)$ is a solution in $C^0_{-\sigma}(\mathbf{R},n)$ (see (15)) of the equation

$$\dot y=D(t,0)\,y+C(t,0)\,x_\varepsilon(t,0)+g_\varepsilon(\alpha_0,y_0(t),0). \tag{53}$$

Then it follows from (52) and (12) that

$$x_\varepsilon(t,0)=x_\varepsilon(0,0)+p(t,\alpha_0).$$

Next, differentiating (9) with respect to α, we see that

$$\int_{-\infty}^{\infty}\psi_\alpha^*(t,\alpha_0)\,g_x+\psi^*(t,\alpha_0)[g_{xx}+g_{xy}\,y_{0\alpha}(t,\alpha_0)]\,\mathrm{d}t=0.$$

So in the last equation for $\Delta_1'(\alpha_0)$ we can replace $p(t, \alpha_0)$ by $x_\varepsilon(t, 0)$ to get

$$\Delta_1'(\alpha_0) = \int_{-\infty}^{\infty} \psi_\alpha^*(t, \alpha_0)[g_\varepsilon + g_x x_\varepsilon(t, 0)]\, dt$$

$$+ \int_{-\infty}^{\infty} \psi^*(t, \alpha_0)[g_{x\varepsilon} + g_{y\varepsilon}\, y_{0\alpha}(t, \alpha_0) + g_{xx} x_\varepsilon(t, 0)$$

$$+ g_{xy}\, y_{0\alpha}(t, \alpha_0)\, x_\varepsilon(t, 0) + g_x\, p_\alpha(t, \alpha_0)]\, dt.$$

Then, differentiating

$$\dot{\psi}^*(t, \alpha) = -\psi^*(t, \alpha)\, g_y(\alpha, y_0(t, \alpha), 0)$$

with respect to α and using (53), we find that

$$\begin{aligned}
\frac{d}{dt}[\psi_\alpha^*(t, \alpha_0)\, y_\varepsilon(t, 0)] &= -[\psi_\alpha^*(t, \alpha_0)\, g_y + \psi^*(t, \alpha_0)\, g_{xy} \\
&\quad + \psi^*(t, \alpha_0)\, g_{yy}\, y_{0\alpha}(t, \alpha_0)]\, y_\varepsilon(t, 0) + \psi_\alpha^*(t, \alpha_0)\, \dot{y}_\varepsilon(t, 0) \\
&= -\psi^*(t, \alpha_0)[g_{xy} + g_{yy}\, y_{0\alpha}(t, \alpha_0)]\, y_\varepsilon(t, 0) \\
&\quad + \psi_\alpha^*(t, \alpha_0)[\dot{y}_\varepsilon(t, 0) - D(t, 0)\, y_\varepsilon(t, 0)] \\
&= -\psi^*(t, \alpha_0)[g_{xy} + g_{yy}\, y_{0\alpha}(t, \alpha_0)]\, y_\varepsilon(t, 0) \\
&\quad + \psi_\alpha^*(t, \alpha_0)[g_x x_\varepsilon(t, 0) + g_\varepsilon].
\end{aligned}$$

Since $\psi_\alpha^*(t, \alpha_0)\, y_\varepsilon(t, 0)$ tends to 0 exponentially as $t \to \pm\infty$, the integral of the last expression vanishes. Hence

$$\begin{aligned}
\Delta_1'(\alpha_0) &= \int_{-\infty}^{\infty} \psi^*(t, \alpha_0)[g_{xy}\, y_\varepsilon(t, 0) + g_{yy}\, y_{0\alpha}(t, \alpha_0)\, y_\varepsilon(t, 0) + g_{x\varepsilon} \\
&\quad + g_{y\varepsilon}\, y_{0\alpha}(t, \alpha_0) + g_{xx} x_\varepsilon(t, 0) \\
&\quad + g_{xy}\, y_{0\alpha}(t, \alpha_0)\, x_\varepsilon(t, 0) + g_x\, p_\alpha(t, \alpha_0)]\, dt \\
&= \int_{-\infty}^{\infty} \psi^*(t)[g_{xx} x_\varepsilon(t, 0) + g_{xy}\, y_\varepsilon(t, 0) + g_{x\varepsilon} \\
&\quad + \{g_{xy} x_\varepsilon(t, 0) + g_{yy}\, y_\varepsilon(t, 0) + g_{y\varepsilon}\}\, y_{0\alpha_0}(t, \alpha_0) + g_x\, p_\alpha(t, \alpha_0)]\, dt \\
&= \int_{-\infty}^{\infty} \psi^*(t)[C_\varepsilon(t, 0) + D_\varepsilon(t, 0)\, y_{0\alpha}(t, \alpha_0) + C(t, 0)\, p_\alpha(t, \alpha_0)]\, dt.
\end{aligned} \tag{54}$$

Thus, combining (51) and (54), we find that

$$h_1'(0)\,\xi - \Delta_1'(\alpha_0)\,\xi$$
$$= (\mu_0^* \xi) \int_{-\infty}^{\infty} \psi^*(t)\{D_\varepsilon(t,0)\,\dot{y}_0(t) + C(t,0)\,f(\alpha_0, y_0(t), 0)]\}\,dt.$$

Finally, differentiating (53) with respect to t, we see that $\dot{y}_\varepsilon(t,0)$ is a C^1 solution of

$$\dot{y} = D(t,0)\,y + C(t,0)\,\dot{x}_\varepsilon(t,0) + D_\varepsilon(t,0)\,\dot{y}_0(t) \tag{55}$$

in $C^0_{-\sigma}(\mathbf{R}, n)$. Thus, using (52),

$$\int_{-\infty}^{+\infty} \psi^*(t)[C(t,0)\,f(\alpha_0, y_0(t), 0) + D_\varepsilon(t,0)\,\dot{y}_0(t)]\,dt = 0 \tag{56}$$

and so

$$h_1'(0) = \Delta_1'(\alpha_0), \tag{57}$$

which is not zero in this case. Thus we have transverse intersection of $\mathscr{M}^{cs}$ and $\mathscr{M}^{cu}$ in the Cherry case, when $\varepsilon \neq 0$ is sufficiently small.

3.3. *Duffing Case*

In this case we have

$$g(x, y, \varepsilon) = G(y) + \varepsilon^2 h(x, y, \varepsilon).$$

Thus, from (24) we get:

$$\begin{aligned} C(t,0) &= 0 \qquad & C_\varepsilon(t,0) &= 0, \\ D(t,0) &= G'(y_0(t)) \qquad & D_\varepsilon(t,0) &= 0. \end{aligned} \tag{58}$$

Then from (42), (44), (57) and since $\Delta_1'(\alpha_0) = 0$ (see the introduction), we have

$$h_1(0) = 0, \qquad h_2(0) = 0, \qquad h_1'(0) = 0.$$

Although it is not strictly necessary, we first show that $h_2'(0) = 0$. From (40), (36) and (37) we get

$$\begin{aligned} h_2'(0) &= \psi^* L'(0)(0, \phi_0, \phi_0) + \psi^* L(0)(0, w_+'(0), w_-'(0)) \\ &= \psi^* L'(0)(0, \phi_0, \phi_0). \end{aligned} \tag{59}$$

Using (47) and (58) we see at once that

$$h_2'(0) = 0.$$

So we look at $h_1''(0)$. From (39), (36), (47) and (58), we obtain:

$$\begin{aligned} h_1''(0)\,\xi &= \psi^* L''(0)(\xi, H_+(0)\,\xi, H_-(0)\,\xi) + 2\psi^* L'(0)(0, H'_+(0)\,\xi, H'_-(0)\,\xi) \\ &= \psi^* L''(0)(\xi, H_+(0)\,\xi, H_-(0)\,\xi) \\ &= \int_{-\infty}^{+\infty} \psi^*(t)\{C_{\varepsilon\varepsilon}(t,0)\,\xi + D_{\varepsilon\varepsilon}(t,0)\,y^{\pm}(t,\xi,H_{\pm}(0)\,\xi,0)\}\,\mathrm{d}t. \end{aligned}$$

Now from the fact that in this case $y_0(t,\alpha) = y_0(t)$ is independent of α, we deduce that the function $y(t,\xi)$ defined in (48) satisfies, because of (49),

$$y(t,\xi) = (\mu_0^*\xi)\,\dot{y}_0(t).$$

Then, using (48):

$$h_1''(0)\,\xi = \int_{-\infty}^{+\infty} \psi^*(t)\{C_{\varepsilon\varepsilon}(t,0)\,\xi + D_{\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)(\mu_0^*\xi)\}\,\mathrm{d}t.$$

Next we note that

$$C_{\varepsilon\varepsilon}(t,0) = 2h_x(\alpha_0, y_0(t), 0)$$

and

$$D_{\varepsilon\varepsilon}(t,0) = G''(y_0(t))\,y_{\varepsilon\varepsilon}(t,0) + 2h_y(\alpha_0, y_0(t), 0),$$

and from

$$\dot{y}(t,\varepsilon) = G(y(t,\varepsilon)) + \varepsilon^2 h(x(t,\varepsilon), y(t,\varepsilon), \varepsilon)$$

we get

$$\begin{aligned} \ddot{y}_{\varepsilon\varepsilon}(t,0) - G'(y_0(t))\,\dot{y}_{\varepsilon\varepsilon}(t,0) &= \{G''(y_0(t))\,y_{\varepsilon\varepsilon}(t,0) + 2h_y(\alpha_0, y_0(t), 0)\}\,\dot{y}_0(t) \\ &= D_{\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t) \end{aligned}$$

which implies:

$$\int_{-\infty}^{+\infty} \psi^*(t)\,D_{\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)\,\mathrm{d}t = 0.$$

Hence:

$$
\begin{aligned}
h_1''(0) &= \int_{-\infty}^{+\infty} \psi^*(t)\, C_{\varepsilon\varepsilon}(t, 0)\, \mathrm{d}t \\
&= 2 \int_{-\infty}^{+\infty} \psi^*(t)\, h_x(\alpha_0, y_0(t), 0)\, \mathrm{d}t \\
&= 2\Delta_2'(\alpha_0).
\end{aligned} \tag{60}
$$

Since in this case $\Delta_2'(\alpha_0) \neq 0$, the transversality of the intersection of $\mathscr{M}^{cs}$ and $\mathscr{M}^{cu}$ in the Duffing case follows.

4. TRANSVERSE INTERSECTION OF $\mathscr{M}^{cs}$ AND $\mathscr{M}^{u}$

In this section we continue to assume that, for the three cases considered in this paper (Kurland–Levi, Cherry, Duffing) the corresponding Melnikov function has a simple zero at a point $\alpha = \alpha_0$. The difference from the previous section consists in that here we will study the problem of the transverse intersection of $\mathscr{M}^{cs}$ and $\mathscr{M}^{u}$ along the solution $(x(t, \varepsilon), y(t, \varepsilon))$. We will show that if

$$\Delta_i'(\alpha_0)\, f(\alpha_0, v_-(\alpha_0), 0) \neq 0,$$

then $\mathscr{M}^{cs}$ and $\mathscr{M}^{u}$ intersect transversally along $(x(t, \varepsilon), y(t, \varepsilon))$.

This transversality is equivalent to showing that system (16) has no non-trivial solutions $(x(t), y(t))$ that satisfy the following estimates:

$$
\begin{aligned}
&|x(t)|, |y(t)| \leqslant Ce^{\beta t} \qquad \text{if} \quad t \leqslant 0 \\
&|x(t)|, |y(t)| \leqslant Ce^{\sigma t} \qquad \text{if} \quad t \geqslant 0.
\end{aligned} \tag{61}
$$

Now such a solution also satisfies

$$|x(t)|, |y(t)| \leqslant Ce^{\sigma |t|} \qquad \text{for all } t$$

and so, from Lemma 2 and the Remark after it,

$$x(t) = x^{\pm}(t, \xi, \zeta_{\pm}, \varepsilon), \qquad y(t) = y^{\pm}(t, \xi, \zeta_{\pm}, \varepsilon)$$

for $t \geqslant 0$ (resp. $t \leqslant 0$), where

$$x(0) = \xi, \qquad P_+\, y(0) = \zeta_+, \qquad (\mathbf{I} - P_-)\, y(0) = \zeta_-$$

and the equations

$$\zeta_\pm = H_\pm(\varepsilon)\,\xi + \omega_\pm(\varepsilon)\,\lambda + \lambda\phi_0, \qquad h_1(\varepsilon)\,\xi + h_2(\varepsilon)\,\lambda = 0 \tag{62}$$

hold for some real λ. However, from Lemma 1 we know that any solution $(x(t), y(t))$ of system (17) that satisfies the first inequality in (61) is a solution of the fixed point equation (20). Thus:

$$\xi = x(0) = \varepsilon \int_{-\infty}^{0} A(t, \varepsilon)\, x^-(t, \zeta_-, \varepsilon) + B(t, \varepsilon)\, y^-(t, \zeta_-, \varepsilon)\, \mathrm{d}t. \tag{63}$$

Using linearity, let

$$\xi = \varepsilon\tilde{\xi}(\varepsilon)\,\zeta_-$$

be the right hand side of (63), where we note that

$$\tilde{\xi}(0) = \int_{-\infty}^{0} B(t, 0)\, Y(t)(\mathbf{I} - P_-)\, \mathrm{d}t.$$

Then we have the two equations

$$\xi = \varepsilon\tilde{\xi}(\varepsilon)\,\zeta_-, \quad \zeta_- = H_-(\varepsilon)\,\xi + \omega_-(\varepsilon)\,\lambda + \lambda\phi_0$$

from which it follows that

$$\xi = \varepsilon\tilde{\xi}(\varepsilon)\, H_-(\varepsilon)\,\xi + \varepsilon\tilde{\xi}(\varepsilon)[\omega_-(\varepsilon) + \phi_0]\,\lambda$$

and so if ε is sufficiently small we can solve for ξ to get

$$\xi = \varepsilon[\mathbf{I} - \varepsilon\tilde{\xi}(\varepsilon)\, H_-(\varepsilon)]^{-1}\, \tilde{\xi}(\varepsilon)[\omega_-(\varepsilon) + \phi_0]\,\lambda = \varepsilon\bar{\xi}(\varepsilon)\,\lambda, \tag{64}$$

where

$$\begin{aligned}
\bar{\xi}(0) &= \int_{-\infty}^{0} B(t, 0)\, Y(t)(\mathbf{I} - P_-)\,\phi_0\, \mathrm{d}t \\
&= \int_{-\infty}^{0} f_y(\alpha_0, y_0(t), 0)\, \dot{y}_0(t)\, \mathrm{d}t \\
&= f(\alpha_0, y_0(0), 0) - f(\alpha_0, v_-(\alpha_0), 0).
\end{aligned}$$

It follows from (62) and (64) that

$$[\varepsilon h_1(\varepsilon)\, \bar{\xi}(\varepsilon) + h_2(\varepsilon)]\,\lambda = 0.$$

Now if $\varepsilon h_1(\varepsilon)\,\bar{\xi}(\varepsilon)+h_2(\varepsilon)\neq 0$, then $\lambda=0$. Then it follows from (64) and (62) that $\xi=0$, $\zeta_+=0$, $\zeta_-=0$ and so $(x(t), y(t))=(0,0)$ for all t and the transversality follows.

Hence the transversality will follow if we can show that

$$T(\varepsilon):=\varepsilon h_1(\varepsilon)\,\bar{\xi}(\varepsilon)+h_2(\varepsilon)\neq 0 \tag{65}$$

for $\varepsilon\neq 0$ sufficiently small. Again we will study the above condition separately for the different cases considered in this paper.

4.1. *Kurland–Levi Case*

From $h_2(0)=0$ (see (42)), which implies $T(0)=0$, we are led to multiply $T(\varepsilon)$ of (65) by ε^{-1} and take the limit for $\varepsilon\to 0$. We obtain:

$$T'(0)=h_1(0)\,\bar{\xi}(0)+h_2'(0).$$

We already know that

$$\bar{\xi}(0)=f(\alpha_0, y_0(0), 0)-f(\alpha_0, v_-(\alpha_0), 0). \tag{66}$$

Next we calculate $h_2'(0)$. From (59) and (47),

$$h_2'(0)=\int_{-\infty}^{\infty}\psi^*(t)[C(t,0)\,x_\varepsilon^{\pm}(t,0,\phi_0,0)+D_\varepsilon(t,0)\,y^{\pm}(t,0,\phi_0,0)]\,\mathrm{d}t. \tag{67}$$

Differentiating the equation

$$\dot{x}=\varepsilon[A(t,\varepsilon)\,x+B(t,\varepsilon)\,y]$$

with respect to ε, and using $x^{\pm}(0,\xi,\zeta_{\pm},\varepsilon)=\xi$ and (29), we see that $x_\varepsilon^{\pm}(t,0,\phi_0,0)$ satisfies:

$$\dot{x}=B(t,0)\,\dot{y}_0(t)$$
$$x(0)=0$$

from which we get:

$$x_\varepsilon^{\pm}(t,0,\phi_0,0)=\int_0^t B(s,0)\,\dot{y}_0(s)\,\mathrm{d}s=f(\alpha_0, y_0(t), 0)-f(\alpha_0, y_0(0), 0).$$

Then, from (29), (56), (67) and (45), we get

$$h_2'(0)=-\int_{-\infty}^{+\infty}\psi^*(t)\,C(t,0)\,\mathrm{d}t\,f(\alpha_0, y_0(0), 0)=-h_1(0)\,f(\alpha_0, y_0(0), 0) \tag{68}$$

and hence, using (66) and (45):

$$h_1(0)\,\bar{\xi}(0)+h_2'(0)=-h_1(0)\,f(\alpha_0, v_-(\alpha_0), 0)=-\Delta_0'(\alpha_0)\,f(\alpha_0, v_-(\alpha_0), 0).$$

Hence, for $\varepsilon \neq 0$ sufficiently small, $\mathscr{M}^{cs}$ and $\mathscr{M}^{u}$ intersect transversally along $(x(t,\varepsilon), y(t,\varepsilon))$ if

$$\Delta_0'(\alpha_0)\,f(\alpha_0, v_-(\alpha_0), 0) \neq 0.$$

4.2. *Cherry Case*

From (42), (46) and (68) we have

$$h_1(0)=h_2(0)=h_2'(0)=0.$$

Thus $T'(0)=0$ and so to study the transversality in this case we multiply $T(\varepsilon)=\varepsilon h_1(\varepsilon)\,\bar{\xi}(\varepsilon)+h_2(\varepsilon)$ by ε^{-2} and take the limit for $\varepsilon \to 0$, obtaining

$$\tfrac{1}{2}T''(0)=h_1'(0)\,\bar{\xi}(0)+\tfrac{1}{2}h_2''(0).$$

If we can show that this quantity is nonzero, then $T(\varepsilon)\neq 0$ for $\varepsilon \neq 0$ sufficiently small and the transversality follows.

From the Cherry case studied in Sec. 3.2 we know that

$$h_1'(0)=\Delta_1'(\alpha_0).$$

From (66) we obtain then:

$$h_1'(0)\,\bar{\xi}(0)=\Delta_1'(\alpha_0)[f(\alpha_0, y_0(0), 0)-f(\alpha_0, v_-(\alpha_0), 0)]. \tag{69}$$

Next we calculate $h_2''(0)$. To this end, let

$$\begin{aligned} w_1^{\pm}(t,\varepsilon)&=x^{\pm}(t,0,\omega_{\pm}(\varepsilon)+\phi_0,\varepsilon),\\ w_2^{\pm}(t,\varepsilon)&=y^{\pm}(t,0,\omega_{\pm}(\varepsilon)+\phi_0,\varepsilon). \end{aligned} \tag{70}$$

From (40), (70) and the definition of $L(\varepsilon)$, we see that

$$h_2(\varepsilon)=\int_{-\infty}^{\infty}\psi^*(t)\{C(t,\varepsilon)\,w_1^{\pm}(t,\varepsilon)+[D(t,\varepsilon)-D(t,0)]\,w_2^{\pm}(t,\varepsilon)\}\,dt,$$

where $\psi(t)=\psi(t,\alpha_0)$. So, differentiating twice and using the facts that $w_1^{\pm}(t,0)=0$ and $w_2^{\pm}(t,0)=\dot{y}_0(t)$, we find that

$$\begin{aligned} h_2''(0)=\int_{-\infty}^{\infty}\psi^*(t)[&2C_\varepsilon(t,0)\,w_{1\varepsilon}^{\pm}(t,0)+C(t,0)\,w_{1\varepsilon\varepsilon}^{\pm}(t,0)\\ &+2D_\varepsilon(t,0)\,w_{2\varepsilon}^{\pm}(t,0)+D_{\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)]\,dt. \end{aligned} \tag{71}$$

We now calculate the derivatives $w^{\pm}_{1\varepsilon}(t, 0)$, $w^{\pm}_{2\varepsilon}(t, 0)$. First from (30) with $\xi = 0$, $\zeta_{\pm} = \omega_{\pm}(\varepsilon) + \phi_0$, we get

$$w_2^+(0, \varepsilon) - w_2^-(0, \varepsilon) = \omega_+(\varepsilon) - \omega_-(\varepsilon) - L(\varepsilon)(0, \omega_+(\varepsilon) + \phi_0, \omega_-(\varepsilon) + \phi_0).$$

However, we know from our solution $\zeta_{\pm} = H_{\pm}(\varepsilon)\,\xi + \omega_{\pm}(\varepsilon)\,\lambda + \lambda\phi_0$ of (34) with $\xi = 0$ and $\lambda = 1$ that

$$\begin{aligned}\omega_+(\varepsilon) - \omega_-(\varepsilon) = {} & L(\varepsilon)(0, \omega_+(\varepsilon) + \phi_0, \omega_-(\varepsilon) + \phi_0) \\ & - \psi^* L(\varepsilon)(0, \omega_+(\varepsilon) + \phi_0, \omega_-(\varepsilon) + \phi_0) \cdot \psi.\end{aligned}$$

Hence, referring to (40), we see that

$$w_2^+(0, \varepsilon) - w_2^-(0, \varepsilon) = -\psi^* L(\varepsilon)(0, \omega_+(\varepsilon) + \phi_0, \omega_-(\varepsilon) + \phi_0) \cdot \psi = -h_2(\varepsilon)\,\psi. \tag{72}$$

Note next that $(x(t), y(t)) = (w^{\pm}_{1\varepsilon}(t, 0), w^{\pm}_{2\varepsilon}(t, 0))$ is a solution of

$$\begin{aligned}\dot{x} &= B(t, 0)\,\dot{y}_0(t)\\ \dot{y} &= D(t, 0)\,y + C(t, 0)\,x + D_\varepsilon(t, 0)\,\dot{y}_0(t)\end{aligned}$$

in $C^0_{-2\sigma}(\mathbf{R}_{\pm}, n+m)$ (see Lemma 2) such that

$$x(0) = 0, \qquad P_+\,y(0) = \omega'_+(0), \qquad (\mathbf{I} - P_-)\,y(0) = \omega'_-(0).$$

In particular,

$$w^{\pm}_{1\varepsilon}(t, 0) = \int_0^t B(s, 0)\,\dot{y}_0(s)\,\mathrm{d}s = \dot{x}_\varepsilon(t, 0) - f(\alpha_0, y_0(0), 0) \tag{73}$$

where we have used (52). Now, from $h_2'(0) = 0$ and (72), we get

$$w^+_{2\varepsilon}(0, 0) = w^-_{2\varepsilon}(0, 0),$$

that is,

$$w_{2\varepsilon}(t, 0) = \begin{cases} w^+_{2\varepsilon}(t, 0) & \text{if} \quad t \geqslant 0,\\ w^-_{2\varepsilon}(t, 0) & \text{if} \quad t \leqslant 0\end{cases}$$

is a solution of

$$\dot{y} = D(t, 0)\,y + C(t, 0)[\dot{x}_\varepsilon(t, 0) - f(\alpha_0, y_0(0), 0)] + D_\varepsilon(t, 0)\,\dot{y}_0(t) \tag{74}$$

in $C^0_{-2\sigma}(\mathbf{R}, n)$. Now, we recall that $y_\varepsilon(t, 0)$ is a solution of equation (53) in $C^0_{-\sigma}(\mathbf{R}, n)$ and that $y_{0\alpha}(t, \alpha)$ is a bounded solution of (8). So, differentiating (53) with respect to t, we see that $\dot{y}_\varepsilon(t, 0) - y_{0\alpha}(t, \alpha_0)\, f(\alpha_0, y_0(0), 0)$ is another solution of (74) in $C^0_{-\sigma}(\mathbf{R}, n)$ and hence $\mu_0 \in \mathbf{R}$ exists such that:

$$w_{2\varepsilon}(t, 0) = \dot{y}_\varepsilon(t, 0) - y_{0\alpha}(t, \alpha_0)\, f(\alpha_0, y_0(0), 0) + \mu_0 \dot{y}_0(t). \tag{75}$$

Next we see that $x(t) = w^{\pm}_{1\varepsilon\varepsilon}(t, 0)$ are solutions of

$$\dot{x} = 2\{A(t, 0)[\dot{x}_\varepsilon(t, 0) - f(\alpha, y_0(0), 0)] + B(t, 0)\, w_{2\varepsilon}(t, 0) + B_\varepsilon(t, 0)\, \dot{y}_0(t)\}$$

such that $x(0) = 0$. So

$$\begin{aligned} w^{\pm}_{1\varepsilon\varepsilon}(t, 0) = 2 \int_0^t & A(s, 0)[\dot{x}_\varepsilon(s, 0) - f(\alpha_0, y_0(0), 0)] \\ & + B(s, 0)\, w_{2\varepsilon}(s, 0) + B_\varepsilon(s, 0)\, \dot{y}_0(s)\, \mathrm{d}s. \end{aligned} \tag{76}$$

Then note from

$$\dot{x}(t, \varepsilon) = \varepsilon f(x(t, \varepsilon), y(t, \varepsilon), \varepsilon)$$

that

$$\ddot{x}(t, \varepsilon) = \varepsilon\{A(t, \varepsilon)\, \dot{x}(t, \varepsilon) + B(t, \varepsilon)\, \dot{y}(t, \varepsilon)\}$$

and so

$$\begin{aligned} \ddot{x}_\varepsilon(t, \varepsilon) = A(t, \varepsilon)\, \dot{x}(t, \varepsilon) + B(t, \varepsilon)\, \dot{y}(t, \varepsilon) + \varepsilon\{& A_\varepsilon(t, \varepsilon)\, \dot{x}(t, \varepsilon) \\ & + A(t, \varepsilon)\, \dot{x}_\varepsilon(t, \varepsilon) + B_\varepsilon(t, \varepsilon)\, \dot{y}(t, \varepsilon) + B(t, \varepsilon)\, \dot{y}_\varepsilon(t, \varepsilon)\}, \end{aligned}$$

and

$$\ddot{x}_{\varepsilon\varepsilon}(t, 0) = 2\{A(t, 0)\, \dot{x}_\varepsilon(t, 0) + B_\varepsilon(t, 0)\, \dot{y}_0(t) + B(t, 0)\, \dot{y}_\varepsilon(t, 0)\}.$$

Next, referring to (12) for the definition of $p(t, \alpha)$, we see that

$$\dot{p}_\alpha(t, \alpha_0) = A(t, 0) + B(t, 0)\, y_{0\alpha}(t, \alpha_0).$$

Then, using these relations and also (75), we obtain from (76):

$$\begin{aligned} w^{\pm}_{1\varepsilon\varepsilon}(t,0) &= 2\int_0^t \tfrac{1}{2}\ddot{x}_{\varepsilon\varepsilon}(s,0) - \dot{p}_\alpha(s,\alpha_0)\, f(\alpha_0, y_0(0), 0)\,\mathrm{d}s \\ &\quad + 2\mu_0[f(\alpha_0, y_0(t), 0) - f(\alpha_0, y_0(0), 0)] \\ &= \dot{x}_{\varepsilon\varepsilon}(t,0) - \dot{x}_{\varepsilon\varepsilon}(0,0) - 2[p_\alpha(t,\alpha_0) - p_\alpha(0,\alpha_0)]\, f(\alpha_0, y_0(0), 0) \\ &\quad + 2\mu_0[f(\alpha_0, y_0(t), 0) - f(\alpha_0, y_0(0), 0)]. \end{aligned} \tag{77}$$

Then, using (71), (73) and (75), and writing

$$w^{\pm}_{1\varepsilon\varepsilon}(t,0) = w_{1\varepsilon\varepsilon}(t,0),$$

we find that

$$\begin{aligned} h''_2(0) = \int_{-\infty}^{\infty} \psi^*(t)\{ & C(t,0)\, w_{1\varepsilon\varepsilon}(t,0) + 2C_\varepsilon(t,0)[\dot{x}_\varepsilon(t,0) - f(\alpha_0, y_0(0), 0)] \\ & + 2D_\varepsilon(t,0)[\dot{y}_\varepsilon(t,0) - y_{0\alpha}(t,\alpha_0)\, f(\alpha_0, y_0(0), 0)] \\ & + D_{\varepsilon\varepsilon}(t,0)\, \dot{y}_0(t) + 2\mu_0 D_\varepsilon(t,0)\, \dot{y}_0(t)\}\,\mathrm{d}t. \end{aligned}$$

Now, differentiating

$$\dot{y}(t,\varepsilon) = g(x(t,\varepsilon), y(t,\varepsilon), \varepsilon)$$

twice with respect to ε we get

$$\begin{aligned} \dot{y}_{\varepsilon\varepsilon}(t,0) &= g_x x_{\varepsilon\varepsilon}(t,0) + g_y\, y_{\varepsilon\varepsilon}(t,0) + g_{xx} x_\varepsilon(t,0)\, x_\varepsilon(t,0) + 2g_{xy} x_\varepsilon(t,0)\, y_\varepsilon(t,0) \\ &\quad + g_{yy}\, y_\varepsilon(t,0)\, y_\varepsilon(t,0) + 2g_{x\varepsilon} x_\varepsilon(t,0) + 2g_{y\varepsilon}\, y_\varepsilon(t,0) + g_{\varepsilon\varepsilon}, \end{aligned}$$

where $g_x = g_x(\alpha_0, y_0(t), 0)$, etc.. Then, differentiating with respect to t,

$$\begin{aligned} \ddot{y}_{\varepsilon\varepsilon}(t,0) &= g_y \dot{y}_{\varepsilon\varepsilon}(t,0) + g_x \dot{x}_{\varepsilon\varepsilon}(t,0) \\ &\quad + 2[g_{xx} x_\varepsilon(t,0) + g_{xy}\, y_\varepsilon(t,0) + g_{x\varepsilon}]\, \dot{x}_\varepsilon(t,0) \\ &\quad + 2[g_{xy} x_\varepsilon(t,0) + g_{yy}\, y_\varepsilon(t,0) + g_{y\varepsilon}]\, \dot{y}_\varepsilon(t,0) \\ &\quad + [g_{xy} x_{\varepsilon\varepsilon}(t,0) + g_{xyx} x_\varepsilon(t,0)\, x_\varepsilon(t,0) + g_{yy}\, y_{\varepsilon\varepsilon}(t,0) \\ &\quad + 2g_{xyy} x_\varepsilon(t,0)\, y_\varepsilon(t,0) + g_{yyy}\, y_\varepsilon(t,0)\, y_\varepsilon(t,0) \\ &\quad + 2g_{xy\varepsilon} x_\varepsilon(t,0) + 2g_{yy\varepsilon}\, y_\varepsilon(t,0) + g_{y\varepsilon\varepsilon}]\, \dot{y}_0(t) \\ &= D(t,0)\, \dot{y}_{\varepsilon\varepsilon}(t,0) + C(t,0)\, \dot{x}_{\varepsilon\varepsilon}(t,0) + 2C_\varepsilon(t,0)\, \dot{x}_\varepsilon(t,0) \\ &\quad + 2D_\varepsilon(t,0)\, \dot{y}_\varepsilon(t,0) + D_{\varepsilon\varepsilon}(t,0)\, \dot{y}_0(t). \end{aligned}$$

So,

$$\int_{-\infty}^{\infty} \psi^*(t)[C(t,0)\,\dot{x}_{\varepsilon\varepsilon}(t,0)+2C_\varepsilon(t,0)\,\dot{x}_\varepsilon(t,0)$$
$$+2D_\varepsilon(t,0)\,\dot{y}_\varepsilon(t,0)+D_{\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)]\,\mathrm{d}t=0$$

and then:

$$\begin{aligned} h_2''(0) &= \int_{-\infty}^{\infty} \psi^*(t)\,C(t,0)[w_{1\varepsilon\varepsilon}(t,0)-\dot{x}_{\varepsilon\varepsilon}(t,0)]\,\mathrm{d}t \\ &\quad -2\int_{-\infty}^{\infty} \psi^*(t)[C_\varepsilon(t,0)+D_\varepsilon(t,0)\,y_{0\alpha}(t,\alpha_0)]\,\mathrm{d}t\cdot f(\alpha_0,y_0(0),0) \\ &\quad +2\mu_0\int_{-\infty}^{\infty} \psi^*(t)\,D_\varepsilon(t,0)\,\dot{y}_0(t)\,\mathrm{d}t. \end{aligned} \tag{78}$$

From (78) and (77) we see that the coefficient of μ_0 in $h_2''(0)$ is

$$2\int_{-\infty}^{\infty} \psi^*(t)\{C(t,0)[f(\alpha_0,y_0(t),0)-f(\alpha_0,y_0(0),0)]+D_\varepsilon(t,0)\,\dot{y}_0(t)\}\,\mathrm{d}t,$$

which is zero because of $h_1(0)=0$ and (56).

Then, using (78), (77), $h_1(0)=\int_{-\infty}^{\infty}\psi^*(t)\,C(t,0)\,\mathrm{d}t=0$ and (54), we find that

$$\begin{aligned} h_2''(0) &= -2\int_{-\infty}^{\infty} \psi^*(t)\{C(t,0)[p_\alpha(t,\alpha_0)-p_\alpha(0,\alpha_0)]\,f(\alpha_0,y_0(0),0) \\ &\qquad +[C_\varepsilon(t,0)+D_\varepsilon(t,0)\,y_{0\alpha}(t,\alpha_0)]\,f(\alpha_0,y_0(0),0)\}\,\mathrm{d}t \\ &= -2\Delta_1'(\alpha_0)\,f(\alpha_0,y_0(0),0). \end{aligned} \tag{79}$$

Finally, putting (69) and (79) together, we get

$$\begin{aligned} \tfrac{1}{2}T''(0) &= h_1'(0)\,\bar{\xi}(0)+\tfrac{1}{2}h_2''(0) \\ &= \Delta_1'(\alpha_0)[f(\alpha_0,y_0(0),0)-f(\alpha_0,v_-(\alpha_0),0)] \\ &\quad -\Delta_1'(\alpha_0)\,f(\alpha_0,y_0(0),0) \\ &= -\Delta_1'(\alpha_0)\,f(\alpha_0,v_-(\alpha_0),0). \end{aligned}$$

So if

$$\Delta_1'(\alpha_0)\,f(\alpha_0,v_-(\alpha_0),0)\neq 0,$$

$T(\varepsilon)\neq 0$ for $\varepsilon\neq 0$ sufficiently small and the transversality follows.

4.3. *Duffing Case*

Note first that in this case it follows from (53), (58) and $g(x, y, \varepsilon) = G(y) + \varepsilon^2 h(x, y, \varepsilon)$, that $y_\varepsilon(t, 0)$ is a multiple of $\dot{y}_0(t)$ since it is a solution of $\dot{y} = D(t, 0)\, y$ in $C^0_{-\sigma}(\mathbf{R}, n)$.

Next from the Duffing case of the previous section and (79) we get

$$h_1(0) = h_2(0) = h_1'(0) = h_2'(0) = h_2''(0) = 0.$$

Now for transversality we need

$$T(\varepsilon) = \varepsilon h_1(\varepsilon)\, \bar{\xi}(\varepsilon) + h_2(\varepsilon) \neq 0.$$

In this case we have:

$$T(0) = h_2(0) = 0,$$
$$T'(0) = h_1(0)\, \bar{\xi}(0) + h_2'(0) = 0$$
$$\tfrac{1}{2} T''(0) = h_1'(0)\, \bar{\xi}(0) + \tfrac{1}{2} h_2''(0) = 0.$$

So we look at

$$\tfrac{1}{6} T'''(0) = \tfrac{1}{2} h_1''(0)\, \bar{\xi}(0) + \tfrac{1}{6} h_2'''(0).$$

Using (60) and (66), we have

$$\tfrac{1}{2} h_1''(0)\, \bar{\xi}(0) = \Delta_2'(\alpha_0)[f(\alpha_0, y_0(0), 0) - f(\alpha_0, v_-, 0)], \tag{80}$$

where, in this case, $v_- = v_-(\alpha)$ is independent of α.

Next we calculate $h_2'''(0)$. From the Cherry case we know that

$$h_2(\varepsilon) = \int_{-\infty}^{\infty} \psi^*(t)[C(t, \varepsilon)\, w_1^\pm(t, \varepsilon) + (D(t, \varepsilon) - D(t, 0))\, w_2^\pm(t, \varepsilon)]\, \mathrm{d}t,$$

where $w_1^\pm(t, \varepsilon)$ and $w_2^\pm(t, \varepsilon)$ are defined in (70). Moreover, from equation (72) and $h_2(0) = h_2'(0) = h_2''(0) = 0$, we get:

$$w_2^+(0, 0) = w_2^-(0, 0), \qquad w_{2\varepsilon}^+(0, 0) = w_{2\varepsilon}^-(0, 0), \qquad w_{2\varepsilon\varepsilon}^+(0, 0) = w_{2\varepsilon\varepsilon}^-(0, 0). \tag{81}$$

Using $w_1^\pm(t, 0) = 0$, $w_2^\pm(t, 0) = \dot{y}_0(t)$ and (58), we obtain:

$$h_2'''(0) = \int_{-\infty}^{+\infty} \psi^*(t)\{3C_{\varepsilon\varepsilon}(t, 0)\, w_{1\varepsilon}^\pm(t, 0) + D_{\varepsilon\varepsilon\varepsilon}(t, 0)\, \dot{y}_0(t)$$
$$+ 3D_{\varepsilon\varepsilon}(t, 0)\, w_{2\varepsilon}^\pm(t, 0)\}\, \mathrm{d}t.$$

Now, equations (73) and (75), together with the fact that $y_\varepsilon(t,0)$ is a multiple of $\dot{y}_0(t)$ and the independence of $y_0(t,\alpha)$ with respect to α in this case give (with perhaps a different μ_0):

$$w^{\pm}_{1\varepsilon}(t,0)=f(\alpha_0, y_0(t), 0)-f(\alpha_0, y_0(0), 0), \qquad w^{\pm}_{2\varepsilon}(t,0)=\mu_0\dot{y}_0(t).$$

Now, in view of (81) and (58) and using $w^{\pm}_1(t,0)=0$, $w^{\pm}_2(t,0)=\dot{y}_0(t)$, the function

$$w_{2\varepsilon\varepsilon}(t,0)=\begin{cases} w^{+}_{2\varepsilon\varepsilon}(t,0) & \text{if } \ t\geqslant 0,\\ w^{-}_{2\varepsilon\varepsilon}(t,0) & \text{if } \ t\leqslant 0\end{cases}$$

is a C^1 solution of

$$\dot{y}=D(t,0)\,y+D_{\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)$$

in $C^0_{-3\sigma}(\mathbf{R}, n)$ (see Lemma 2). Thus

$$\int_{-\infty}^{+\infty}\psi^*(t)\,D_{\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)\,\mathrm{d}t=0$$

and then, using $C_{\varepsilon\varepsilon}(t,0)=2h_x(\alpha_0, y_0(t), 0)$ and the definition of $\Delta_2(\alpha)$:

$$\begin{aligned} h_2'''(0)&=\int_{-\infty}^{+\infty}\psi^*(t)\{3C_{\varepsilon\varepsilon}(t,0)[f(\alpha_0, y_0(t), 0)-f(\alpha_0, y_0(0), 0)]\\ &\quad+D_{\varepsilon\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)\}\,\mathrm{d}t\\ &=-6\Delta_2'(\alpha_0)\,f(\alpha_0, y_0(0), 0)\\ &\quad+\int_{-\infty}^{+\infty}\psi^*(t)\{3C_{\varepsilon\varepsilon}(t,0)\,f(\alpha_0, y_0(t), 0)+D_{\varepsilon\varepsilon\varepsilon}(t,0)\dot{y}_0(t)\}\,\mathrm{d}t. \end{aligned} \tag{82}$$

Now, differentiating the equation

$$\dot{y}(t,\varepsilon)=G(y(t,\varepsilon))+\varepsilon^2 h(x(t,\varepsilon), y(t,\varepsilon), \varepsilon)$$

with respect to ε three times, we obtain

$$\dot{y}_{\varepsilon\varepsilon\varepsilon}(t,0)=G'(y_0(t))\,y_{\varepsilon\varepsilon\varepsilon}(t,0)+6h_x(\alpha_0, y_0(t), 0)\,x_\varepsilon(t,0)+6h_\varepsilon(\alpha_0, y_0(t), 0)$$

and then, differentiating with respect to t,

$$\begin{aligned}\ddot{y}_{\varepsilon\varepsilon\varepsilon}(t,0)-G'(y_0(t))\,\dot{y}_{\varepsilon\varepsilon\varepsilon}(t,0)\\ &= G''(y_0(t))\,y_{\varepsilon\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)+6h_x(\alpha_0,y_0(t),0)\,\dot{x}_\varepsilon(t,0)\\ &\quad +6[h_{xy}(\alpha_0,y_0(t),0)\,x_\varepsilon(t,0)+h_{y\varepsilon}(\alpha_0,y_0(t),0)]\,\dot{y}_0(t)\\ &= D_{\varepsilon\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)+3C_{\varepsilon\varepsilon}(t,0)\,f(\alpha_0,y_0(t),0).\end{aligned}$$

As a consequence

$$\int_{-\infty}^{+\infty}\psi^*(t)\{3C_{\varepsilon\varepsilon}(t,0)\,f(\alpha_0,y_0(t),0)+D_{\varepsilon\varepsilon\varepsilon}(t,0)\,\dot{y}_0(t)\}\,\mathrm{d}t=0$$

and so, from (82),

$$h_2'''(0)=-6\Delta_2'(\alpha_0)\,f(\alpha_0,y_0(0),0). \tag{83}$$

Finally, equations (80) and (83) together give

$$\tfrac{1}{6}T'''(0)=\tfrac{1}{2}h_1''(0)\,\bar{\xi}(0)+\tfrac{1}{6}h_2'''(0)=-\Delta_2'(\alpha_0)\,f(\alpha_0,v_-,0).$$

Hence if

$$\Delta_2'(\alpha_0)\,f(\alpha_0,v_-,0)\neq 0,$$

$T(\varepsilon)\neq 0$ for $\varepsilon\neq 0$ sufficiently small and the transversality follows.

APPENDIX

Here we show Lemmas 1 and 2 of Section 2. Since their proofs are quite similar we will only give the proof of Lemma 1 in some detail and just indicate the changes that have to be made to prove Lemma 2. Let $\beta\in((r+3)\,\sigma,\delta)$. For any $x\in C^0_\beta(\mathbf{R}_-,m)$, $\zeta_-\in\mathcal{N}P_-$, consider the equation:

$$\begin{aligned}\dot{y}&=D(t,\varepsilon)\,y+C(t,\varepsilon)\,x(t)\\ (\mathbf{I}-P_-)\,y(0)&=\zeta_-.\end{aligned} \tag{84}$$

From the fact that $\mathcal{R}P_-(\varepsilon)$ is independent of ε (assumption (H1)), we see that $(\mathbf{I}-P_-(\varepsilon))\,P_-=0$ and hence the second equality in (84) can be written

$$(\mathbf{I}-P_-(\varepsilon))\,y(0)=(\mathbf{I}-P_-(\varepsilon))\,\zeta_-.$$

Next, using again assumption (H1) about the exponential dichotomy of $\dot{y}=D(t,\varepsilon)\,y$, we see that (84) has a unique solution $y(t)=y^{-}(t,x,\zeta_{-},\varepsilon)$ in $C^0_\beta(\mathbf{R}_{-},n)$ that satisfies the equation

$$
\begin{aligned}
y(t)=&\,Y(t,\varepsilon)(\mathbf{I}-P_{-}(\varepsilon))\,\zeta_{-}\\
&+\int_{-\infty}^{t} Y(t,\varepsilon)\,P_{-}(\varepsilon)\,Y^{-1}(s,\varepsilon)\,C(s,\varepsilon)\,x(s)\,\mathrm{d}s\\
&-\int_{t}^{0} Y(t,\varepsilon)(\mathbf{I}-P_{-}(\varepsilon))\,Y^{-1}(s,\varepsilon)\,C(s,\varepsilon)\,x(s)\,\mathrm{d}s.
\end{aligned}
\tag{85}
$$

This means that

$$y^{-}(t,x,\zeta_{-},\varepsilon)=[Y_1^{-}(\varepsilon)\,x](t)+[Y_2^{-}(\varepsilon)\,\zeta_{-}](t),$$

where $Y_1^{-}(\varepsilon)\in\mathscr{L}(C^0_\beta(\mathbf{R}_{-},m),C^0_\beta(\mathbf{R}_{-},n))$ and $Y_2^{-}(\varepsilon)\in\mathscr{L}(\mathscr{N}P_{-},C^0_\beta(\mathbf{R}_{-},n))$ are defined by:

$$
\begin{aligned}
[Y_1^{-}(\varepsilon)\,x](t)=&\int_{-\infty}^{t} Y(t,\varepsilon)\,P_{-}(\varepsilon)\,Y^{-1}(s,\varepsilon)\,C(s,\varepsilon)\,x(s)\,\mathrm{d}s\\
&-\int_{t}^{0} Y(t,\varepsilon)(\mathbf{I}-P_{-}(\varepsilon))\,Y^{-1}(s,\varepsilon)\,C(s,\varepsilon)\,x(s)\,\mathrm{d}s,
\end{aligned}
$$

and

$$[Y_2^{-}(\varepsilon)\,\zeta_{-}](t)=Y(t,\varepsilon)(\mathbf{I}-P_{-}(\varepsilon))\,\zeta_{-}.$$

The following property holds:

(a) $\|Y_1^{-}(\varepsilon)\|\leqslant 2kN(\delta-\beta)^{-1}$, and $\|Y_2^{-}(\varepsilon)\|\leqslant k$.

We now show that for $i=1,2$, $Y_i^{-}(\varepsilon)$ satisfy the following condition:

(b) $Y_i^{-}(\varepsilon)$, $i=1,2$ are C^k_{lip} in ε, $k\leqslant r+1$, as maps into the Banach spaces $\mathscr{L}(C^0_\beta(\mathbf{R}_{-},m),C^0_{\beta-(k+1)\sigma}(\mathbf{R}_{-},n))$ and $\mathscr{L}(\mathscr{N}P_{-},C^0_{\beta-(k+1)\sigma}(\mathbf{R}_{-},n))$ respectively. More precisely, in the case of $Y_1^{-}(\varepsilon)$, for any positive integers k, j such that $k+j\leqslant r+1$, the k-th ε-derivative of $Y_1^{-}(\varepsilon)$ can be extended to a linear map from $C^0_{\beta-j\sigma}(\mathbf{R}_{-},m)$ into the space $C^0_{\beta-(k+j)\sigma}(\mathbf{R}_{-},n)$ which is Lipschitz-continuous in ε when considered into $C^0_{\beta-(k+j+1)\sigma}(\mathbf{R}_{-},n)$.

We begin with Lipschitz continuity ($k=j=0$). Let

$$\tilde{y}(t):=[(Y_1^{-}(\varepsilon+h)-Y_1^{-}(\varepsilon))\,x](t).$$

Then $\tilde{y} \in C^0_\beta(\mathbf{R}_-, n) \subset C^0_{\beta-\sigma}(\mathbf{R}_-, n)$ satisfies the following equation

$$\begin{aligned}\dot{y} &= D(t, \varepsilon)\, y + [D(t, \varepsilon+h) - D(t, \varepsilon)][Y_1^-(\varepsilon+h)\, x](t)\\ &\quad + [C(t, \varepsilon+h) - C(t, \varepsilon)]\, x(t)\\ (\mathbf{I} - P_-)\, y(0) &= 0\end{aligned}$$

and then:

$$\begin{aligned}\|\tilde{y}\|_{\beta-\sigma} \leqslant 2k(\delta-\beta+\sigma)^{-1} \{&\|[D(\cdot, \varepsilon+h) - D(\cdot, \varepsilon)][Y_1^-(\varepsilon+h)\, x](\cdot)\|_{\beta-\sigma}\\ &+ \|[C(\cdot, \varepsilon+h) - C(\cdot, \varepsilon)]\, x(\cdot)\|_{\beta-\sigma}\}.\end{aligned}$$

Now, from condition (H2) with $k=0$, and (a) we obtain

$$\begin{aligned}&\|[D(\cdot, \varepsilon+h) - D(\cdot, \varepsilon)][Y_1^-(\varepsilon+h)\, x](\cdot)\|_{\beta-\sigma}\\ &\quad = \sup_{t \leqslant 0} |[D(t, \varepsilon+h) - D(t, \varepsilon)][Y_1^-(\varepsilon+h)\, x](t)|\, e^{-(\beta-\sigma)\, t}\\ &\quad \leqslant \sup_{t \leqslant 0} |D(t, \varepsilon+h) - D(t, \varepsilon)|\, e^{\sigma t} \cdot \sup_{t \leqslant 0} |[Y_1^-(\varepsilon+h)\, x](t)|\, e^{-\beta t}\\ &\quad \leqslant 2kN^2\, |h| \cdot (\delta-\beta)^{-1}\, \|x\|_\beta,\end{aligned}$$

and, similarly,

$$\|[C(\cdot, \varepsilon+h) - C(\cdot, \varepsilon)]\, x(\cdot)\|_{\beta-\sigma} \leqslant N\, |h|\, \|x\|_\beta.$$

Thus,

$$\|(Y_1^-(\varepsilon+h) - Y_1^-(\varepsilon))\, x\|_{\beta-\sigma} \leqslant 2kN(\delta-\beta+\sigma)^{-1}\, \{2kN(\delta-\beta)^{-1} + 1\}\, |h|\, \|x\|_\beta$$

and so $Y_1^-(\varepsilon)$ is Lipschitz-continuous in ε as a map into the space $\mathscr{L}(C^0_\beta(\mathbf{R}_-, m), C^0_{\beta-\sigma}(\mathbf{R}_-, n))$. Similarly, the function $[(Y_2^-(\varepsilon+h) - Y_2^-(\varepsilon))\, \zeta_-](t)$ is a solution in $C^0_\beta(\mathbf{R}_-, m)$ of

$$\begin{aligned}\dot{y} &= D(t, \varepsilon)\, y + [D(t, \varepsilon+h) - D(t, \varepsilon)][Y_2^-(\varepsilon+h)\, \zeta_-](t)\\ (\mathbf{I} - P_-)\, y(0) &= 0\end{aligned}$$

and the same method as above shows that $Y_2^-(\varepsilon)$ is Lipschitz-continuous in ε as a map into $\mathscr{L}(\mathscr{N} P_-, C^0_{\beta-\sigma}(\mathbf{R}_-, n)$.

Now we show that the derivatives $Y_{i\varepsilon}^-(\varepsilon) = DY_i^-(\varepsilon)$, $i = 1, 2$, exist and satisfy estimates similar to $Y_i^-(\varepsilon)$. Taking the formal derivative of Eq. (84)

with respect to ε, we see that $y(t) = [Y^-_{1\varepsilon}(\varepsilon)\, x](t)$ (if it exists) has to satisfy:

$$\begin{aligned} &\dot{y} = D(t, \varepsilon)\, y + C_\varepsilon(t, \varepsilon)\, x(t) + D_\varepsilon(t, \varepsilon)[Y^-_1(\varepsilon)\, x](t) \\ &(\mathbf{I} - P_-)\, y(0) = 0. \end{aligned} \tag{86}$$

Similarly, $y(t) = [Y^-_{2\varepsilon}(\varepsilon)\, \zeta_-](t)$ (if it exists) has to satisfy:

$$\begin{aligned} &\dot{y} = D(t, \varepsilon)\, y + D_\varepsilon(t, \varepsilon)[Y^-_2(\varepsilon)\, \zeta_-](t) \\ &(\mathbf{I} - P_-)\, y(0) = 0. \end{aligned} \tag{87}$$

Now consider equation (86) (a similar argument applies to (87)). The same reasoning as in the previous part (note that $|C_\varepsilon(t, \varepsilon)| \leqslant N e^{-\sigma t}$ and, similarly, for $D_\varepsilon(t, \varepsilon)$) shows that, for any $x \in C^0_{\beta - j\sigma}(\mathbf{R}_-, m)(1 \leqslant j \leqslant r)$, equation (86) has a unique solution in $C^0_{\beta-(j+1)\sigma}(\mathbf{R}_-, n)$, say $\bar{y}(t)$. By linearity in x, we can write

$$\bar{y}(t) = [Z(\varepsilon)\, x](t),$$

where $Z(\varepsilon)$ is in $\mathscr{L}(C^0_{\beta-j\sigma}(\mathbf{R}_-, m), C^0_{\beta-(j+1)\sigma}(\mathbf{R}_-, n))$ and can be shown to be Lipschitz-continuous in ε as a map into $\mathscr{L}(C^0_{\beta-j\sigma}(\mathbf{R}_-, m), C^0_{\beta-(j+2)\sigma}(\mathbf{R}_-, n))$. Property (b) with $k = r = 1$ will be proved if we can show that $Z(\varepsilon)$ is in fact the ε-derivative of $Y^-_1(\varepsilon)$ as a map into $\mathscr{L}(C^0_\beta(\mathbf{R}_-, m), C^0_{\beta-2\sigma}(\mathbf{R}_-, n))$. To this end, let $x \in C^0_\beta(\mathbf{R}_-, m)$ and set

$$y(t) = [(Y^-_1(\varepsilon + h) - Y^-_1(\varepsilon))\, x](t) - [Z(\varepsilon)\, x](t)\, h.$$

Then $y \in C^0_{\beta-2\sigma}(\mathbf{R}_-, n)$ satisfies:

$$\begin{aligned} \dot{y} = D(t, \varepsilon)\, y &+ [C(t, \varepsilon + h) - C(t, \varepsilon) - C_\varepsilon(t, \varepsilon)\, h]\, x(t) \\ &+ [D(t, \varepsilon + h) - D(t, \varepsilon) - D_\varepsilon(t, \varepsilon)\, h][Y^-_1(\varepsilon)\, x](t) \\ &+ h D_\varepsilon(t, \varepsilon)[(Y^-_1(\varepsilon + h) - Y^-_1(\varepsilon))\, x](t) \\ (\mathbf{I} - P_-)\, y(0) &= 0. \end{aligned}$$

Thus, using (a), we obtain

$$\begin{aligned} \|y\|_{\beta-2\sigma} \leqslant 2k(\delta - \beta + 2\sigma)^{-1} \{ &\|C(\cdot, \varepsilon + h) - C(\cdot, \varepsilon) - C_\varepsilon(\cdot, \varepsilon)\, h\|_{-2\sigma}\, \|x\|_\beta \\ &+ \|[D(\cdot, \varepsilon + h) - D(\cdot, \varepsilon) - D_\varepsilon(\cdot, \varepsilon)\, h]\|_{-2\sigma} \cdot \|[Y^-_1(\varepsilon)\, x]\|_\beta \\ &+ |h|\, \|D_\varepsilon(\cdot, \varepsilon)\|_{-\sigma}\, \|[(Y^-_1(\varepsilon + h) - Y^-_1(\varepsilon))\, x]\|_{\beta-\sigma} \} \end{aligned}$$

$$\begin{aligned}&\leqslant 2k(\delta-\beta+2\sigma)^{-1}\,[\tfrac{1}{2}N\,|h|^2\,\|x\|_\beta+\tfrac{1}{2}N\,|h|^2\cdot 2k(\delta-\beta)^{-1}\,\|x\|_\beta\\&\quad+N\,|h|\cdot 2kN(\delta-\beta+\sigma)^{-1}\,\{2kN(\delta-\beta)^{-1}+1\}\,|h|\,\|x\|_\beta]\\&=\tilde{C}\,|h|^2\,\|x\|_\beta\end{aligned}$$

which gives the desired conclusion. More arguments of a similar nature show the statement in (b).

Then in order to find solutions of the system

$$\begin{aligned}\dot{x}&=\varepsilon A(t,\varepsilon)\,x+\varepsilon B(t,\varepsilon)\,y\\\dot{y}&=C(t,\varepsilon)\,x+D(t,\varepsilon)\,y\end{aligned}$$

that belong to $C^0_\beta(\mathbf{R}_-, n+m)$, and satisfy

$$(\mathbf{I}-P_-)\,y(0)=\zeta_-,$$

we have to study the problem of existence of solutions x in $C^0_\beta(\mathbf{R}_-, m)$ of

$$\dot{x}=\varepsilon[A(t,\varepsilon)\,x+B(t,\varepsilon)\{[Y_1^-(\varepsilon)\,x](t)+[Y_2^-(\varepsilon)\,\zeta_-](t)\}]. \tag{88}$$

If $x(t)$ is any such solution, then for $b\leqslant t\leqslant 0$,

$$x(t)=x(b)+\varepsilon\int_b^t A(s,\varepsilon)\,x(s)+B(s,\varepsilon)\{[Y_1^-(\varepsilon)\,x](s)+[Y_2^-(\varepsilon)\,\zeta_-](s)\}\,ds.$$

Letting $b\to-\infty$, we obtain

$$x(t)=\varepsilon\int_{-\infty}^t A(s,\varepsilon)\,x(s)+B(s,\varepsilon)\{[Y_1^-(\varepsilon)\,x](s)+[Y_2^-(\varepsilon)\,\zeta_-](s)\}\,ds,$$

the above integrals converging because of the properties of $A(t,\varepsilon)$, $B(t,\varepsilon)$ and $Y_1^-(\varepsilon)$ and $Y_2^-(\varepsilon)$. So for any fixed $x\in C^0_\beta(\mathbf{R}_-, m)$ let

$$\tilde{x}(t)=\varepsilon\int_{-\infty}^t A(s,\varepsilon)\,x(s)+B(s,\varepsilon)\{[Y_1^-(\varepsilon)\,x](s)+[Y_2^-(\varepsilon)\,\zeta_-](s)\}\,ds. \tag{89}$$

From assumption (H2), and property (a) of $Y_1^-(\varepsilon)$ and $Y_2^-(\varepsilon)$, we get at once:

$$\|\tilde{x}\|_\beta\leqslant\varepsilon\beta^{-1}N\{[1+2k(\delta-\beta)^{-1}]\,\|x\|_\beta+k\,|\zeta_-|\}. \tag{90}$$

Similarly, if $x_1, x_2\in C^0_\beta(\mathbf{R}_-, m)$ and $\tilde{x}_1, \tilde{x}_2$ are defined as in (89), we obtain:

$$\|\tilde{x}_2-\tilde{x}_1\|_\beta\leqslant\varepsilon\beta^{-1}N[1+2k(\delta-\beta)^{-1}]\,\|x_2-x_1\|_\beta.$$

Thus, assuming ε is so small that

$$a := \varepsilon\beta^{-1}N[1+2k(\delta-\beta)^{-1}] < 1,$$

the map $x \mapsto \tilde{x}$ is a contraction in $C^0_\beta(\mathbf{R}_-, m)$ and hence has a unique fixed point $x(t) = x(t, \zeta_-, \varepsilon)$ which is linear in ζ_- for any fixed ε. Hence we can write

$$x(t, \zeta_-, \varepsilon) = [X^-(\varepsilon)\,\zeta_-](t),$$

where $X^-(\varepsilon) \in \mathscr{L}(\mathscr{N}P_-, C^0_\beta(\mathbf{R}_-, m))$ and satisfies

$$\|X^-(\varepsilon)\| \leqslant C\,|\varepsilon|, \qquad C = [(1-a)\,\beta]^{-1}\,kN,$$

this last estimate coming from (90) when we replace $x(t)$ with $[X^-(\varepsilon)\,\zeta_-](t)$.

We now show that the the following holds:

(c) For $|\varepsilon|$ less that some ε_0, $X^-(\varepsilon)$ is smooth in ε as in the statement of Lemma 1.

Again we start with Lipschitz-continuity. If we let

$$x(t) = [(X^-(\varepsilon+h) - X^-(\varepsilon))\,\zeta_-](t),$$

then $x \in C^0_\beta(\mathbf{R}_-, m) \subset C^0_{\beta-\sigma}(\mathbf{R}_-, m)$ solves

$$\begin{aligned}
\dot{x} = \varepsilon\{&A(t, \varepsilon)\,x + [A(t, \varepsilon+h) - A(t, \varepsilon)][X^-(\varepsilon+h)\,\zeta_-](t)\\
&+ [B(t, \varepsilon+h) - B(t, \varepsilon)]\; y^-(t, X^-(\varepsilon+h)\,\zeta_-, \zeta_-, \varepsilon+h)\\
&+ B(t, \varepsilon)[y^-(t, X^-(\varepsilon+h)\,\zeta_-, \zeta_-, \varepsilon+h) - y^-(t, X^-(\varepsilon)\,\zeta_-, \zeta_-, \varepsilon)]\\
&+ h\{A(t, \varepsilon+h)[X^-(\varepsilon+h)\,\zeta_-](t)\\
&+ B(t, \varepsilon+h)\; y^-(t, X^-(\varepsilon+h)\,\zeta_-, \zeta_-, \varepsilon+h)\}\}.
\end{aligned}$$

Thus, using the same method as above,

$$\begin{aligned}
&(1-a)\,\|x\|_{\beta-\sigma}\\
&\quad\leqslant \varepsilon\{\|A(\cdot, \varepsilon+h) - A(\cdot, \varepsilon)\|_{-\sigma}\,\|X^-(\varepsilon+h)\,\zeta_-\|_\beta\\
&\qquad + \|B(\cdot, \varepsilon+h) - B(\cdot, \varepsilon)\|_{-\sigma}\,\|y^-(\cdot, X^-(\varepsilon+h)\,\zeta_-, \zeta_-, \varepsilon+h)\|_\beta\\
&\qquad + N\,\|y^-(\cdot, X^-(\varepsilon+h)\,\zeta_-, \zeta_-, \varepsilon+h) - y^-(\cdot, X^-(\varepsilon)\,\zeta_-, \zeta_-, \varepsilon)\|_{\beta-\sigma}\\
&\qquad + |h|\,N[\|X(\varepsilon+h)\,\zeta_-\|_{\beta-\sigma} + \|y^-(\cdot, X^-(\varepsilon+h)\,\zeta_-, \zeta_-, \varepsilon+h)\|_{\beta-\sigma}]\}\\
&\quad= O(|h|)\,\|\zeta_-\|
\end{aligned}$$

because of the Lipschitz-continuity of $Y_j^-(\varepsilon)$ and the assumptions on $A(t, \varepsilon)$, $B(t, \varepsilon)$. Thus $X^-(\varepsilon)$ is Lipschitz-continuous as a map into the Banach space $\mathscr{L}(\mathscr{N}P_-, C^0_{\beta-\sigma}(\mathbf{R}_-, m))$.

Next, taking the formal derivative of equation (88) with respect to ε we see that if $X_\varepsilon^-(\varepsilon)$ exists, then

$$x(t) = [X_\varepsilon^-(\varepsilon)\,\zeta_-](t)$$

has to satisfy:

$$\begin{aligned} \dot{x}(t) = \varepsilon\{&A(t, \varepsilon)\,x(t) + B(t, \varepsilon)[Y_1^-(\varepsilon)\,x](t)\} \\ &+ [A(t, \varepsilon) + \varepsilon A_\varepsilon(t, \varepsilon)][X^-(\varepsilon)\,\zeta_-](t) \\ &+ [B(t, \varepsilon) + \varepsilon B_\varepsilon(t, \varepsilon)][Y_1^-(\varepsilon)\,X^-(\varepsilon)\,\zeta_- + Y_2^-(\varepsilon)\,\zeta_-](t) \\ &+ \varepsilon B(t, \varepsilon)[Y_{1\varepsilon}^-(\varepsilon)\,X^-(\varepsilon)\,\zeta_- + Y_{2\varepsilon}^-(\varepsilon)\,\zeta_-](t). \end{aligned} \tag{91}$$

The same arguments used to show the existence of $X^-(\varepsilon)\,\zeta_-$ show that (91) has a unique solution, $x(t) = [\bar{X}^-(\varepsilon)\,\zeta_-](t)$, linear in ζ_-, that belongs to $C^0_{\beta-\sigma}(\mathbf{R}_-, m)$. Apart from some additional technical difficulties, the proof that $\bar{X}^-(\varepsilon)$ is in fact the derivative with respect to ε of $X^-(\varepsilon)$ as a map into $\mathscr{L}(C^0_\beta(\mathbf{R}_-, m), C^0_{\beta-2\sigma}(\mathbf{R}_-, m))$ follows the same lines as the proof of the smoothness of $Y_j^-(\varepsilon)$ and so we omit it. The higher derivatives can be treated similarly.

To conclude the proof of Lemma 1 we define $Y^-(\varepsilon)$ in $\mathscr{L}(\mathscr{N}P_-, C^0_\beta(\mathbf{R}_-, n))$ by

$$Y^-(\varepsilon) = Y_1^-(\varepsilon)\,X^-(\varepsilon) + Y_2^-(\varepsilon).$$

Then $([X^-(\varepsilon)\,\zeta_-](t), [Y^-(\varepsilon)\,\zeta_-](t))$ is the unique solution of (17) in the space $C^0_\beta(\mathbf{R}_-, m+n)$ satisfying (19). Next we outline the proof that $Y^-(\varepsilon)$ is smooth as in the statement of Lemma 1. We have for example

$$\begin{aligned} &\|[Y_1^-(\varepsilon+h)\,X^-(\varepsilon+h) - Y_1^-(\varepsilon)\,X^-(\varepsilon)]\,\zeta_-\|_{\beta-\sigma} \\ &\quad\leqslant \|[Y_1^-(\varepsilon+h) - Y_1^-(\varepsilon)]\,X^-(\varepsilon+h)\,\zeta_-\|_{\beta-\sigma} \\ &\qquad + \|Y_1^-(\varepsilon)[X^-(\varepsilon+h) - X^-(\varepsilon)]\,\zeta_-\|_{\beta-\sigma} \\ &\quad = O(|h|)\,|\zeta_-| \end{aligned}$$

because of the properties of $Y_1^-(\varepsilon)$ and $X^-(\varepsilon)$. Thus $Y_1^-(\varepsilon)\,X^-(\varepsilon)$ is Lipschitz-continuous in ε as a map into $\mathscr{L}(\mathscr{N}P_-, C^0_{\beta-\sigma}(\mathbf{R}_-, m))$. Since the same holds for $Y_2^-(\varepsilon)$ (see point (b)), we deduce that $Y^-(\varepsilon)$ is Lipschitz-continuous in ε as a map into $\mathscr{L}(\mathscr{N}P_-, C^0_{\beta-\sigma}(\mathbf{R}_-, m))$. As for the ε-derivative of $Y^-(\varepsilon)$ we see that it is given by:

$$Y_\varepsilon^-(\varepsilon) = Y_{1\varepsilon}^-(\varepsilon)\, X^-(\varepsilon) + Y_1^-(\varepsilon)\, X_\varepsilon^-(\varepsilon) + Y_{2\varepsilon}^-(\varepsilon) \in \mathscr{L}(\mathscr{N}P_-, C^0_{\beta-\sigma}(\mathbf{R}_-, n)).$$

In fact we have, for example,

$$\begin{aligned}
&\|[Y_1^-(\varepsilon+h)\, X^-(\varepsilon+h) - Y_1^-(\varepsilon)\, X^-(\varepsilon) \\
&\qquad - hY_{1\varepsilon}^-(\varepsilon)\, X^-(\varepsilon) - hY_1^-(\varepsilon)\, X_\varepsilon^-(\varepsilon)]\, \zeta_-\|_{\beta-2\sigma} \\
&\quad \leqslant \|[Y_1^-(\varepsilon+h) - Y_1^-(\varepsilon) - hY_{1\varepsilon}^-(\varepsilon)]\, X^-(\varepsilon+h)\, \zeta_-\|_{\beta-2\sigma} \\
&\qquad + \|Y_1^-(\varepsilon)[X^-(\varepsilon+h) - X^-(\varepsilon) - hX^-(\varepsilon)]\, \zeta_-\|_{\beta-2\sigma} \\
&\quad = O(|h|^2)\, |\zeta_-|.
\end{aligned}$$

The same method as above applies to show the continuity of $Y_\varepsilon^-(\varepsilon)$.

Finally, we sketch the main differences in the proof of Lemma 2 in the case of negative t. It is not difficult to see that equation (85) gives a solution of (84) in $C^0_{-\sigma}(\mathbf{R}_-, n)$, for any $x \in C^0_{-\sigma}(\mathbf{R}_-, m)$. We write

$$y(t) = y^-(t, x, \zeta_-, \varepsilon) = [Y_1^-(\varepsilon)\, x](t) + [Y_2^-(\varepsilon)\, \zeta_-](t).$$

Then (a) still holds with σ instead of β. As for (b), it is easy to see that it can be replaced with the following:

(b′) Suppose that $(r+3)\,\sigma < \delta$. Then $Y_i^-(\varepsilon)$, $i = 1, 2$ are C^k_{lip} in ε, $k \leqslant r+1$, as maps into $\mathscr{L}(C^0_{-\sigma}(\mathbf{R}_-, m), C^0_{-(k+2)\,\sigma}(\mathbf{R}_-, n))$. More precisely, in the case of $Y_1^-(\varepsilon)$, for any positive integers k, j such that $k+j \leqslant r+1$, the k-th ε-derivative of $Y_1^-(\varepsilon)$ can be extended to a linear map from $C^0_{-j\sigma}(\mathbf{R}_-, m)$ into $C^0_{-(k+j+1)\,\sigma}(\mathbf{R}_-, n)$ which is Lipschitz-continuous when considered into $C^0_{-(k+j+2)\,\sigma}(\mathbf{R}_-, n)$.

So the first part of the proof is almost the same as in the case of exponential decay. Then we proceed in the following way: for ε sufficiently small, the system

$$\begin{aligned}
\dot{x}(t) &= \varepsilon A(t, \varepsilon)\, x(t) + \varepsilon B(t, \varepsilon)\, y^-(t, x(\cdot), \zeta_-, \varepsilon) \\
x(0) &= \xi
\end{aligned}$$

has a unique solution $x^-(t, \xi, \zeta_-, \varepsilon) \in C^0_{-\sigma}(\mathbf{R}_-, m)$ that satisfies the fixed point equation

$$x(t) = \xi + \varepsilon \int_0^t A(s, \varepsilon)\, x(s) + B(s, \varepsilon)\, y^-(s, x(\cdot), \zeta_-, \varepsilon)\, ds.$$

Again the properties of $y^-(s, x(\cdot), \zeta_-, \varepsilon)$ and assumption (H2) imply that $x^-(t, \xi, \zeta_-, \varepsilon) = [X_1(\varepsilon)\,\xi](t) + [X_2(\varepsilon)\,\zeta_-](t)$ is linear in (ξ, ζ_-) and that $X_1(\varepsilon)$ (resp. $X_2(\varepsilon)$) is smooth in ε as in the statement of Lemma 2. The conclusion follows as in the exponential decaying case by taking

$$[Y^-(\varepsilon)(\xi, \zeta_-)](t) = [Y_1^-(\varepsilon)\{X_1(\varepsilon)\,\xi + X_2(\varepsilon)\,\zeta_-\}](t) + [Y_2^-(\varepsilon)\,\zeta_-](t).$$

We conclude this appendix noting that any solution $y(t)$ of (84) is also a solution of the equation

$$\begin{aligned} &\dot{y}(t) - D(t, 0)\, y(t) = C(t, \varepsilon)\, x(t) + [D(t, \varepsilon) - D(t, 0)]\, y(t) \\ &(\mathbf{I} - P_-)\, y(0) = \zeta_- \end{aligned}$$

and then if such a solution belongs to $C^0_\beta(\mathbf{R}_-, n))$ with $\beta \in (-\delta, \delta)$, it satisfies the integral equation

$$\begin{aligned} y(t) = {} & Y(t)\,\zeta_- \\ & + \int_{-\infty}^{t} Y(t)\, P_- Y^{-1}(s)\{C(s, \varepsilon)\, x(s) + [D(s, \varepsilon) - D(s, 0)]\, y(s)\}\, \mathrm{d}s \\ & - \int_{t}^{0} Y(t)(\mathbf{I} - P_-)\, Y^{-1}(s)\{C(s, \varepsilon)\, x(s) + [D(s, \varepsilon) - D(s, 0)]\, y(s)\}\, \mathrm{d}s \end{aligned}$$

because of the exponential dichotomy of the fundamental matrix $Y(t)$ of system $\dot{y}(t) = D(t, 0)\, y(t)$ with exponent δ. This remark validates the fixed point equations (20) and (23). (21) and (22) can be similarly validated.

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